PHYSICAL REVIEW B VOLUME 3, NUMBER 3 1 FEBRUARY 1971

Dynamics of an Antiferromagnet at Low Temperatures: Spin-Wave Damping and Hydrodynamics*

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(Received 14 July 1970)

The Dyson-Maleev boson formulation is used to investigate the dynamical properties of Heisenberg antiferromagnets at long wavelengths and low temperatures. Various regimes for the decay rate of spin waves are found, depending on the relation between the wave vector k, the temperature T, and the anisotropy energy $\hbar\omega_A$, and in all cases the decay rate is much smaller than the spin-wave frequency. This result implies that spin waves are well-defined elementary excitations, which interact weakly at low temperatures and long wavelengths, in contrast to results obtained by previous authors, but in close analogy with the ferromagnetic case. When the long-wavelength limit is taken at fixed temperature, the decay rate $\Gamma_{\vec{k}}^{*}$ is proportional to the square of the frequency $\omega_E\epsilon_{\vec{k}}$, where ω_E is the exchange frequency. In the quantum-mechanical low-temperature limit $(ST \ll T_N)$, we find $\Gamma_{\vec{k}}^{*}=2\omega_E S^{-2}\epsilon_{\vec{k}}^2\tau^3(2\pi)^{-3}(a\,|\ln\tau\,|+a')$ for $\epsilon_{\vec{k}}\ll\tau^3\ll 1$, where $\tau=2k_BT/\hbar\omega_E$, and S is the spin quantum number. In the classical low-temperature limit $(T_N/S\ll T\ll T_N)$, we find $\Gamma_{\vec{k}}=(4\eta/3\pi)\omega_E(T/T_N)^2\epsilon_{\vec{k}}^2$ for $\epsilon_{\vec{k}}\ll 1$. For small uniaxial single-ion anisotropy $[\epsilon_0\sim(2\omega_A/\omega_E)^{1/2}\ll 1]$, we find $\Gamma_0=\frac{3}{2}\omega_E S^{-2}\epsilon_0^2\tau^3(2\pi)^{-3}(a\,|\ln\tau\,|+a'')$ for $\epsilon_0\ll\tau^3\ll 1$. (In these expressions, a, a', η, and a'' are all constants of order unity.) Results are also obtained for other regimes, and for the damping of a spin wave driven off resonance. In each case, the nature and self-consistency of the perturbation expansion are examined in detail. For the isotropic system, the full frequency-dependent transverse spin-correlation functions are calculated in the long-wavelength limit, and are found to agree with the forms previously obtained by hydrodynamic arguments. By a comparison of the two forms, the transport coefficients are determined at low temperatures. Several of the calculations have been performed using the Holstein-Primakoff as well as the Dyson-Maleev representations. The results for observable quantities agree in the two formalisms, except at the longest wavelengths, where the Holstein-Primakoff expressions are not self-consistent in lowest order. Finally, the possibility of experimental verification of the present calculations is briefly discussed.

OUTLINE

I. INTRODUCTION

Spin waves have been studied extensively in magnetic systems, since the early work of Bloch[1] and Holstein and Primakoff[2] on ferromagnets, and its extension to antiferromagnets by Anderson[3] and Kubo.[4] In the first place, spin waves have been considered as elementary excitations from which one can derive the thermodynamic properties of magnetic systems at low temperatures. In the second place, spin waves can be used to calculate various time-dependent properties of magnetic systems, such as dynamic response functions and correlation functions.

One of the most notable examples of a thermodynamic calculation based on spin waves is the work of Dyson.[5] Using a general formalism based on quantum field theory, Dyson showed that spin waves may be used to obtain an asymptotic expansion for the thermodynamic functions of a Heisenberg ferromagnet at low temperatures. Dyson's formalism has also been used to calculate various time-dependent properties of ferromagnets, in particular, the frequency renormalization and damping of the spin-wave pole in correlation functions for various low-temperature regimes.[6-8] A generalization of Dyson's method has been used by one of the authors[9] to calculate various thermodynamic properties of an antiferromagnetic insulator, and also to obtain the renormalization of the spin-wave frequency in that system.

In the present paper, we use this formalism to find expressions for the spin-wave damping[10] and spin-correlation functions, at low temperatures, in both quantum-mechanical and classical Heisenberg antiferromagnets. The correlation functions at long wavelengths are found to have precisely the form predicted earlier by two of the authors,[11,12] on the basis of a macroscopic "hydrodynamic" theory of spin waves[13] analogous to the two-fluid hydrodynamics of liquid helium.[14,15] Specifically, the decay rate of spin waves is predicted to vanish as the square of the frequency, in contrast to the results of various earlier microscopic calculations of spin-wave damping,[16-18] which we believe to be incorrect. One of the aims of this paper is to study the relationship between the microscopic and hydrodynamic theories.

In the hydrodynamic theory,[13] as opposed to the low-temperature microscopic theories, spin waves are found at all temperatures below the Néel point, but they are well defined only for long wavelengths. (The restriction to long wavelengths means that the hydrodynamic spin waves do not contribute significantly to thermodynamic functions, even at low temperatures.) The microscopic calculation gives the limits of validity of hydrodynamics, which the latter theory is itself unable to specify. Moreover, a microscopic calculation of the spin-correlation functions leads to an evaluation at low temperatures of the transport coefficients which enter the hydrodynamic theory as unknown temperature-dependent parameters.

The analogy between magnetic systems and liquid helium, which motivated the development of spin-wave hydrodynamics,[13] may also be carried over to other aspects of spin dynamics. The three levels[19,20] of description of liquid helium – hydrodynamic, phenomenological, and microscopic – have their parallel in a magnetic system, although the emphasis and historical development have been somewhat different. The phenomenological Landau theory of phonons and rotons,[14,15] valid at low temperatures in liquid ^{4}He, is related to the "macroscopic" approach, employed for magnetic systems by Herring and Kittel,[21] Akhiezer, Bar'yakhtar, and Kaganov,[22] and Kaganov and Tsukernik.[23] In liquid helium, the Landau theory has been extremely useful in describing both the thermodynamic and the transport properties up to relatively high temperatures.[24] Moreover, it preceded the microscopic theories and, in fact, yields essentially all interesting physical results more simply than the microscopic approach. The approximate treatment of the Bose gas by Bogoliubov[25] closely parallels the work of Bloch,[1] Holstein and Primakoff,[2] Anderson,[3] and Kubo,[4] whereas the formally exact theory of Beliaev[26] and others[19,27] is analogous to the modern field-theoretic formulations[5,28,29] in the magnetic system. Generally speaking, the microscopic approach has been far more useful in magnetic systems, whereas the phenomenological theories have produced more results in helium. This is because simple models, such as the Heisenberg model which is approximately soluble at low temperatures, provide a rather accurate description of the properties of real magnetic systems. For the Bose system on the other hand, only the dilute gas[25,30] model is approximately soluble microscopically, and its energy spectrum is in lowest order qualitatively different from that of liquid helium. The hydrodynamics and phenomenological theories, however, apply to any interacting Bose liquid.

In the superfluid at low temperatures, it is found[19] that the phonon elementary excitations are very closely related to the hydrodynamic first-sound mode, in that both appear as the dominant pole of the density correlation function.[31] The main difference between the two modes is that first sound is confined to the hydrodynamic low-frequency regime ($\omega \ll \omega_{tr}$), whereas the phonons exist in the high-frequency regime ($\omega \gg \omega_{tr}$) which is often referred to as "collisionless." In passing from one regime to the other at fixed temperature, the velocity is almost unchanged, but the frequency and temperature dependence of the attenuation are different for the two.[31–33] The transition frequency ω_{tr} between the two regimes is the rate of relaxation of the short-wavelength phonons to the local equilibrium state in the presence of a long-wavelength sound wave. For the processes relevant to the attenuation of sound, this relaxation rate is the decay rate Γ_{th} for typical thermally excited phonons. At low temperatures the frequency ω_{tr} vanishes as a high power of the temperature,[33] so that over most of the frequency domain, and, in particular, for thermal frequencies ($\omega \sim k_B T/\hbar$), the density excitations are phonons. As the temperature goes to zero, the hydrodynamic regime becomes vanishingly small in liquid helium.

In contrast to first sound, the hydrodynamic second-sound mode[14,15] does not have any low-temperature elementary excitation associated with it. It exists only for $\omega \ll \omega_{tr}$, where it appears in the correlation functions for the order parameter and for the energy density, and with very small weight in the density correlation function.[19] The second-sound mode disappears completely in the collisionless regime $\omega \gg \omega_{tr}$, which is reached in the limit $T \rightarrow 0$, for any fixed ω.[33] These results follow from both the phenomenological and microscopic theories and are more fully discussed in Refs. 19 and 31–33. The phenomenological[34–36] and microscopic[37,38] treatments of second sound in a solid give results analogous to those for liquid helium.

In the isotropic antiferromagnet the order parameter is coupled to one mode (with two polarizations) at long wavelengths,[39] the spin-wave mode. In the quantum-mechanical low-temperature limit ($k_B T \ll JzS \sim k_B T_N/S$, where J is the exchange constant, S is the spin, T_N is the Néel temperature, and z is the number of nearest neighbors) the relation between "hydrodynamic" and "elementary excitation" spin waves is very similar to the case of first sound in liquid helium.[40] One difference, however, is that from the Born approximation we find that the transition between hydrodynamic and nonhydrodynamic damping occurs at a frequency $\omega_{tr} \sim \hbar^{-1} J \times (k_B T/JzS)^3$. This frequency is much larger than the decay rate Γ_{th} of the thermally excited magnons, which is of order $\hbar^{-1} J (k_B T/JzS)^5$. In the nonhydrodynamic regime, $\omega \gg \omega_{tr}$, the decay rate will have different forms, depending on the ratios of ω to the thermal frequency $k_B T/\hbar$ and of $k_B T$ to the deviation from linearity of the energy spectrum of the incident magnons. In the classical low-temperature regime ($\hbar \rightarrow 0$, $S \rightarrow \infty$, $T_N/S \ll T \ll T_N$), we find that the limiting frequency ω_{tr} for hydrodynamic damping is independent of temperature, and there is no distinction between "hydrodynamic" and "elementary excitation" spin waves, for wavelengths long compared to the lattice spacing.

The calculations in this paper are performed using the boson formalism of Dyson,[5] in the representation proposed by Maleev.[41] For the isotropic model, the lowest Born approximation is shown to yield a self-consistent result for the low-temperature spin-wave damping, in both high-frequency and hydrodynamic regimes, i.e., the results are not changed when decay of the intermediate states is included. As in the case of the thermodynamic calculations,[9] the expansion parameters are z^{-1}, S^{-1}, and $k_B T/JzS$ in the quantum case. Corrections to the lowest Born approximation for the decay rate are of higher order in these small parameters. For classical systems, the corrections are of order T/T_N.[42] The fact that the lowest Born approximation in the Dyson-Maleev formalism yields the correct low-temperature result for arbitrary wavelengths is significant, in view of the apparent singular nature of the magnon-magnon interaction at long wavelengths. Indeed, the "zeroth-order" magnons, obtained in this formalism by a transformation[2,25] from the original Hamiltonian, interact via potentials whose matrix elements occur in the combination

$$\langle 12 | V | 34 \rangle \langle 34 | V | 12 \rangle \sim (1 - \hat{k}_1 \cdot \hat{k}_2)(1 - \hat{k}_3 \cdot \hat{k}_4), \quad (1.1)$$

which remains finite when $\vec{k}_1$, say, goes to zero (here $\hat{k}_1 = \vec{k}_1/|\vec{k}_1|$, etc.). However, when the energies of the four magnons in Eq. (1.1) are related by the condition

$$\epsilon_{\vec{1}} + \epsilon_{\vec{2}} - \epsilon_{\vec{3}} - \epsilon_{\vec{4}} = 0, \qquad (1.2)$$

one may show that the quantity in Eq. (1.1) vanishes linearly for $\epsilon_{\vec{1}} \to 0$. As will be demonstrated in detail in what follows, even when the "energy-shell" condition (1.2) is violated, the matrix elements of interest for calculating physical quantities occur in combinations which vanish when one momentum tends toward zero. This cancellation is shown to persist in higher orders also. Consequently, we conclude that the apparent constancy of the interaction strength as one momentum tends toward zero is a misleading artifact of the formalism. A complete analysis, in which terms are properly grouped, leads to a weaker interaction, in agreement with the intuitive notion that the interaction between magnons should vanish at long wavelengths. We believe that it is the failure to account properly for this detailed behavior, which has led a number of previous authors[16–18] to incorrect expressions.

In the model with uniaxial single-ion anisotropy, which we also treat, we obtain similar results. Indeed, within factors of order unity, one can obtain the damping constant of the uniform mode for the anisotropic system from the results for the isotropic case by replacing the magnon energy $E_{\vec{k}}$ by $(2H_E H_A)^{1/2}$, the energy of the uniform mode in the presence of small anisotropy. One difference between the two cases does emerge, however. Namely, when $E_{\vec{k}} \ll \hbar\Gamma_{\text{th}}$ a self-consistent calculation of the decay rate shows that although the dominant term is unmodified, small changes occur in terms which are of higher order in τ, when the damping of intermediate states is taken into account. Thus, strictly speaking, "hydrodynamic" behavior does not persist for frequencies above Γ_{th}, in contrast to the case of the isotropic system. This transitional behavior disappears as $\epsilon_0 \to 0$, in agreement with the results for the isotropic case. Our results differ as they did in the isotropic case from those of previous authors,[17,18,43–47] for a variety of reasons which we shall discuss later.

Many of the results of the present work may also be obtained by using the boson formalism of Holstein and Primakoff,[2] and expanding the Hamiltonian in powers of the occupation numbers. As is well known, this procedure can lead to incorrect results for small values of the spin S, but the difficulties are avoided if S^{-1} is formally treated as a small parameter and calculations are performed consistently to each order in S^{-1}.[48] It then turns out that for the processes for which the "energy-shell" condition, Eq. (1.2), applies, the matrix elements, and hence the magnon decay rate, agree in the Dyson-Maleev and Holstein-Primakoff formalisms. Off the energy shell, however, the matrix elements in the latter formalism are more singular than in the former, and large corrections to the lowest Born approximation occur for small values of T, k, and S^{-1}, when the damping of intermediate magnons is included. These corrections are of order $(k_B T/JzS)^5/S^2\epsilon_{\vec{k}}$ quantum mechanically, and of order $(T/T_N)^2/\epsilon_{\vec{k}}$ classically, and signify that the Born approximation is not self-consistent at low temperatures in the long-wavelength limit. We have not attempted to find the class of higher-order terms which makes the Holstein-Primakoff answer self-consistent, since this is achieved directly with the Dyson-Maleev formalism. We have checked, however, that where the corrections to the Holstein-Primakoff result are estimated to be small, its physical predictions agree with those obtained using the Dyson-Maleev formalism. This question is discussed in Appendix F.

The primary aim of this work is to explore the formal relationship between microscopic and macroscopic theories of spin waves, and in particular to verify the previously proposed[11,12] hydrodynamic behavior at long wavelengths. As has been remarked elsewhere,[13] the experimental methods for measuring the properties of spin waves accurately are somewhat limited, and the possibilities for verifying hydrodynamics are much less extensive than in superfluid helium. Thus, although we have derived a number of new results in the present work, the possibility of their direct experimental verification does not at present seem very promising, except insofar as they agree with hydrodynamics. The use of the parallel pumping technique[49,50] for verifying some of our results is briefly mentioned, but a detailed discussion requires further calculations. In view of the existence of "two-dimensional" Heisenberg magnets,[51] it would be of interest to see whether such systems display the effects discussed here more prominently.

This paper is organized as follows. In Sec. II, the Dyson-Maleev formalism is introduced, and an attempt is made to spell out the assumptions involved in replacing the spin Hamiltonian by a boson Hamiltonian. Green's functions for the spin system are written in terms of the exact boson Green's functions, and a transformation[2,25] is made to magnon variables, in terms of which actual calculations are carried out.

In Sec. III, the imaginary part of the magnon self-energy is calculated in the quantum-mechanical low-temperature regime, in lowest Born approximation. The incident magnon is assumed to be "on-resonance" ($\hbar\omega = E_{\vec{k}}$), so that this calculation determines the damping of a free spin wave as a function of temperature and wave number. Various regimes must be distinguished, for which the scattering surfaces are different, depending on the relative sizes of the following: (a) the incident energy, (b) the thermal energy, (c) the deviation from linearity of the dispersion relation ("curvature energy") of thermal magnons, and (d) the curvature energy of the incident magnon. In each case, detailed answers are obtained at long wavelengths and low temperatures for a bcc lattice, with an antiferromagnetic interaction between nearest neighbors on opposite sublattices. In particular, when the energy $E_{\vec{k}}$ is less than the thermal curvature energy $[E_{\vec{k}} \ll JzS(k_B T/JzS)^3]$, the damping is found to be proportional to k^2, in agreement with hydrodynamics and in disagreement with previous calculations.

In Sec. IV, the imaginary part of the self-energy is calculated in the hydrodynamic regime $[E_{\vec{k}} \ll JzS \times (k_B T/JzS)^3]$, as a function of ω and k separately, for an incoming magnon whose frequency is not on-resonance ($\hbar\omega \neq E_{\vec{k}}$). In this case, extra diagrams must be included in lowest Born approximation, which vanish when the incoming magnon is on resonance. These terms are then shown to make the lowest Born approximation stable with respect to a self-consistent inclusion of damping in the intermediate states. In Sec. V, the contribution of higher-order diagrams is discussed in some detail, and it is argued that these do not change the k and ω dependence of the damping at long wavelengths. It is furthermore argued, although not proved, that the higher-order terms produce corrections of order $(zS)^{-1}$ and $k_B T/JzS$ to the Born-approximation expression for the coefficient of the k^2 term in the decay rate. From this analysis, it follows that long-wavelength magnons interact very weakly. This conclusion is physically plausible, and in fact enables us to develop a consistent analogy between ferro- and antiferromagnets.

In Sec. VI, the decay rate is calculated for a long-wavelength magnon in the classical low-temperature regime ($T \ll T_N$), obtained by letting $\hbar \to 0$, $S \to \infty$, $J \to 0$, with $k_B T_N \sim JzS^2$ and $\hbar S$ remaining finite. The Born-approximation calculation is slightly more complicated in this case, since the momenta of the intermediate magnons are no longer restricted by the temperature, and the matrix elements must be calculated for arbitrary wave numbers. It turns out, however, that the calculation is possible, and quite similar to the previous case, as long as the incident momentum is small. The classical result once again agrees with hydrodynamics.

In Sec. VII, expressions are found for the spin-correlation functions and a detailed correspondence is made with the hydrodynamic forms; as a result, the transport coefficients are determined at low temperatures. Some general remarks are also made here about the form of the spectral weight of the spin-correlation functions outside the long-wavelength and low-temperature regimes considered in the rest of this paper. A physical argument is given to understand why the transition frequency ω_{tr} for reaching local equilibrium can be larger than the decay rate Γ_{th} of thermal magnons, in contrast to the case of phonons in a crystal[31, 36–38] or in liquid helium.[31–33]

In Sec. VIII, the spin-wave damping is calculated at $k=0$ for the anisotropic model in the quantum low-temperature limit. Generally speaking, these calculations are similar to those given for the isotropic model, and both the results and the regimes can be obtained from those of the isotropic case by everywhere replacing the isotropic magnon energies by the corresponding expressions for the anisotropic system. Here, only the term from the first Born approximation which dominates at low temperatures is self-consistent with respect to the inclusion of damping in intermediate states.

Finally, in Sec. IX, the results of the present calculations are summarized and compared with earlier work. In most cases, previous authors had found a significantly stronger interaction between long-wavelength magnons than is found here. The applicability to experiment is also briefly considered.

A number of detailed calculations are described in the Appendixes. A variety of useful limiting forms of the magnon-magnon interaction in the Dyson-Maleev formalism are given in Appendix A. In Appendix B, the off-diagonal elements of the self-energy matrix are evaluated; these are required for the calculation in Sec. VII of the spin-correlation functions. In Appendix C, a diagrammatic formulation of the vertex functions needed to relate the spin-correlation functions to the usual single-particle boson Green's functions is presented, and the relevant functions are evaluated in lowest Born approximation. In Appendix D, a discussion is presented of the various integrals encountered in the evaluation performed in Sec. IV of off-shell contributions to the self-energy. In Appendix E, it is shown that the spontaneous decay of one magnon into two or more magnons is forbidden by energy conservation in a system in which the only interactions are antiferromagnetic couplings between spins on opposite sublattices.

Appendix F is devoted to the Holstein-Primakoff formalism. In particular, it is shown that in the regime where the Holstein-Primakoff results are self-consistent they lead to spin-correlation functions (but not boson Green's functions) which are identical to those obtained in Sec. VII using the Dyson-Maleev formalism.

Appendixes G, H, and I are concerned with details of the higher-order calculations discussed in Secs. V and VII. In Appendix G, a grouping of diagrams of all orders is constructed which eliminates the spurious strong interaction between magnons, and the resulting dependence of the vertex functions on wave vector and temperature is discussed. In Appendix H, it is shown that the effective interactions obtained in higher order are Hermitian on-resonance (i) for the ferromagnet and (ii) for a particular set of third-order terms in the antiferromagnet. In Appendix I, the spin-correlation functions at long wavelength and low temperature are evaluated to all orders in S^{-1} in terms of vertex functions. In that regime, certain relations between the vertex functions are necessary for the validity of hydrodynamics. These relations are found to hold in low-order perturbation theory.

The main results of this paper have been given previously.[52] In Sec. IX of the present paper, a more extensive and self-contained discussion of our results is given.

II. FORMALISM

In this section, we describe the formal apparatus necessary for the calculations carried out in later sections. Section II A contains a discussion of the problems involved in attempting to reproduce spin kinematics through the use of a boson formalism. In Sec. II B, the application of the Green's-function formalism to the Dyson-Maleev Hamiltonian is described. The spin-correlation functions, which are the physically interesting dynamical quantities, are introduced in Sec. II C. Their relationship to the usual single-particle Green's function is discussed here and in Appendix C. Finally, Sec. II D concerns the perturbation expansion for the decay rate of on-resonance magnons using the "occupation-number" formulation[53,54] of perturbation theory.

A. Justification of Dyson-Maleev Hamiltonian

We wish to study an antiferromagnetic spin system governed by the Hamiltonian

$$\mathcal{H} = 2J\sum_{\langle ij\rangle} \vec{S}_i \cdot \vec{S}_j - D\sum_i (S_i^z)^2 - D\sum_j (S_j^z)^2 - h\sum_i S_i^z + h\sum_j S_j^z , \qquad (2.1)$$

where $\langle ij\rangle$ indicates a sum over pairs of nearest neighbors and the indexes i and j refer to sites on the a ("up") and b ("down") sublattices, respectively, each sublattice consisting of N sites. We choose units such that the positive quantities J and D have dimensions of energy, and the spin operators S_i and S_j are dimensionless, and have magnitude equal to S. The quantity h is an infinitesimal staggered magnetic field necessary to define the direction of orientation of the up and down sublattices, but which otherwise does not enter our calculations.

The Dyson-Maleev transformation gives a correspondence between any operator $\mathcal{O}$ on the Hilbert space of the spin system and an operator $\tilde{\mathcal{O}}$ on a boson Hilbert space.[5] In particular, for the $6N$ spin operators $\vec{S}_i$ and $\vec{S}_j$ we have the following corresponding boson operators[55]:

$$\tilde{S}_i^z = S - a_i^\dagger a_i , \qquad (2.2a)$$

$$\tilde{S}_i^+ = (2S)^{1/2} a_i - (2S)^{-1/2} a_i^\dagger a_i a_i , \qquad (2.2b)$$

$$\tilde{S}_i^- = (2S)^{1/2} a_i^\dagger , \qquad (2.2c)$$

$$\tilde{S}_j^z = -S + b_j^\dagger b_j , \qquad (2.2d)$$

$$\tilde{S}_j^+ = (2S)^{1/2} b_j^\dagger - (2S)^{-1/2} b_j^\dagger b_j^\dagger b_j , \qquad (2.2e)$$

$$\tilde{S}_j^- = (2S)^{1/2} b_j , \qquad (2.2f)$$

where a_i and b_j are boson annihilation operators. The transformed Hamiltonian $\tilde{\mathcal{H}}$ may be obtained by substituting the transformed operators $\tilde{\vec{S}}_i$ and $\tilde{\vec{S}}_j$ for the spin operators in Eq. (2.1). The Hamiltonian $\tilde{\mathcal{H}}$ will involve terms up to fourth order in boson creation and annihilation operators.[56]

Corresponding to a spin operator $\mathcal{O}$ we define the time-dependent operators in the Heisenberg picture as

$$\mathcal{O}(t) = e^{i\mathcal{H}t/\hbar}\, \mathcal{O}\, e^{-i\mathcal{H}t/\hbar} , \qquad (2.3a)$$

$$\tilde{\mathcal{O}}(t) = e^{i\tilde{\mathcal{H}}t/\hbar}\, \tilde{\mathcal{O}}\, e^{-i\tilde{\mathcal{H}}t/\hbar} . \qquad (2.3b)$$

In this paper, we wish to calculate functions of the form[57]

$$\frac{\mathrm{Tr}[\mathcal{O}_1(t_1)\,\mathcal{O}_2(t_2)\cdots\mathcal{O}_n(t_n)\,e^{-\mathcal{H}/k_BT}]}{\mathrm{Tr}(e^{-\mathcal{H}/k_BT})} , \qquad (2.4)$$

where $\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_n$ are spin operators, the times $t_1, t_2, \ldots, t_n$ are on the imaginary axis in the interval $(0, -i\beta)$, and the trace is over the spin Hilbert space. Using the procedure established by Dyson,[5] one can prove the exact relation

$$\mathrm{Tr}[\mathcal{O}_1(t_1)\,\mathcal{O}_2(t_2)\cdots, \ \mathcal{O}_n(t_n)\,e^{-\beta\mathcal{H}}] = \mathrm{Tr}[P\,\tilde{\mathcal{O}}_1(t_1)\,\tilde{\mathcal{O}}_2(t_2)\cdots, \ \tilde{\mathcal{O}}_n(t_n)\,e^{-\beta\tilde{\mathcal{H}}}] , \qquad (2.5)$$

where the trace on the right-hand side of Eq. (2.5) is over the boson Hilbert space, and the operator P is a projection operator which is zero on the "nonphysical" boson states, i.e., the states where one or more lattice sites are each occupied by more than $2S$ bosons. By substituting Eq. (2.5) into the numerator and denominator of Eq. (2.4) we express Green's functions for the spin system in terms of Green's functions for a Bose system with the Hamiltonian $\tilde{\mathcal{H}} = \mathcal{H}_{\mathrm{DM}}$.

In order to evaluate the required averages for the Bose system, it seems to be necessary to make two approximations. First, we expand the Green's functions using a perturbation formalism in which the unperturbed Hamiltonian is quadratic in boson operators and the perturbation is the remaining quartic interaction. Second, we neglect completely the projection operator P.[58] Both of these approximations are correct, in some sense, in the limit when deviations from the boson vacuum state are small.

For the computation of the partition function of a ferromagnet, Dyson has argued that the use of these two approximations would lead to results which are asymptotically correct at low temperatures to all orders in the temperature T. Thus, one can calculate exactly all the contributions to the free energy which vanish more slowly than any given power of T by summing an appropriate set of terms in the perturbation expansion. The corresponding expansions for the Green's functions in the ferromagnet are also expected to be asymptotically exact in this sense, at least if the values of t_1, t_2, $\ldots$, t_n are held fixed on the imaginary axis while T tends to zero. Corrections to the low-temperature results for the free energy, or for the Green's functions, are believed to be roughly of the form $e^{-T_c/T}$ for the ferromagnet,[59] where T_c is approximately equal to the Curie temperature.

The situation for the antiferromagnet is much less clear than for the ferromagnet. Because of the zero-point motion of the antiferromagnet, there is a finite deviation from the Bose vacuum state, i.e., the Néel state, even at 0 °K. Thus one may suspect that there are errors in the perturbation theory even at 0 °K, which may be of order e^{-T_N/T^*}, where T^* is some measure of the zero-point motion. One indication of the magnitude of T^* is the zero-temperature demagnetization, which is known[3,4] to be quite small for three-dimensional systems, even for spin $S=\frac{1}{2}$ and nearest-neighbor interactions. The demagnetization will be smaller still for larger values of the spin, and/or longer-range interactions.[9] It seems clear, therefore, that corrections to the various results of the Dyson-Maleev perturbation formalism will generally be rather small at low temperatures.

In particular, we believe that the lifetimes of the spin waves that we compute are quite accurate at low temperatures, in that corrections are very small in absolute value. We cannot completely rule out the possibility, however, that the true decay rate might have a less rapid variation with k or with T, than the decay rate computed within the Dyson-Maleev formalism. In extreme regimes these corrections would then dominate the Dyson-Maleev results. For example, there might be a decay rate of the form

$$T^2 k^2 e^{-T_N/T^*}, \tag{2.6}$$

which would eventually be larger than the result of order $T^3 k^2 \ln T$, which we find in Sec. III, if the temperature is sufficiently small. Of course, a correction such as in Eq. (2.6) would not be possible in the classical limit, where $T^* \to 0$. In *both* the classical and quantum cases, however, we cannot completely rule out corrections to the decay rate such as

$$k e^{-T_N/T}, \tag{2.7}$$

which would predominate over k^2T^2 or $k^2T^3\ln T$ in the limit $k\to 0$, at any fixed temperature. On the other hand, as we show below, the behavior obtained using the Dyson-Maleev formalism agrees with general hydrodynamic predictions,[13] unlike Eq. (2.7), and therefore we consider the latter behavior extremely unlikely.

B. Green's-Function Formalism with Dyson-Maleev Hamiltonian

We define the spatial Fourier transforms of the boson operators in Eqs. (2.2) by the relations

$$a_{\vec{k}} = N^{-1/2}\sum_i e^{-i\vec{k}\cdot\vec{r}_i} a_i , \tag{2.8a}$$

$$b_{\vec{k}} = N^{-1/2}\sum_j e^{-i\vec{k}\cdot\vec{r}_j} b_j , \tag{2.8b}$$

where $\vec{r}_i$ and $\vec{r}_j$ are the positions of the spins $\vec{S}_i$ and $\vec{S}_j$, respectively. In terms of these operators, the Hamiltonian becomes[60]

$$\mathcal{H}_{\rm DM} = \mathcal{H}^{0\,\prime}_{\rm DM} + V'_{\rm DM} + E'_0 , \tag{2.9}$$

with

$$E'_0 = -2JzS^2N - 2NDS^2 , \tag{2.10a}$$

$$\mathcal{H}^{0\prime}_{\rm DM} = (H_E + H_A)\sum_{\vec{p}} (a^\dagger_{\vec{p}} a_{\vec{p}} + b^\dagger_{\vec{p}} b_{\vec{p}} + \xi\gamma_{\vec{p}} a^\dagger_{\vec{p}} b^\dagger_{-\vec{p}} + \xi\gamma_{\vec{p}} a_{\vec{p}} b_{-\vec{p}}) , \tag{2.10b}$$

$$V'_{\rm DM} = -N^{-1} \sum_{\vec{1}\vec{2}\vec{3}\vec{4}} \delta_{\vec{K}}(\vec{1}+\vec{2}-\vec{3}-\vec{4}) \Big[\frac{H_E}{2S} (2\gamma_{\vec{2}-\vec{4}} a^\dagger_{\vec{1}} b^\dagger_{-\vec{4}} a_{\vec{3}} b_{-\vec{2}} + \gamma_{\vec{2}} a^\dagger_{\vec{1}} b_{-\vec{2}} a_{\vec{3}} a_{\vec{4}} + \gamma_{\vec{3}+\vec{4}-\vec{2}} a^\dagger_{\vec{1}} b^\dagger_{-\vec{3}} b^\dagger_{-\vec{4}} b_{-\vec{2}}) + D(a^\dagger_{\vec{1}} a^\dagger_{\vec{2}} a_{\vec{3}} a_{\vec{4}} + b^\dagger_{\vec{1}} b^\dagger_{\vec{2}} b_{\vec{3}} b_{\vec{4}}) \Big] , \tag{2.10c}$$

where

$$H_A \equiv \hbar\omega_A = D(2S-1) , \tag{2.11a}$$

$$H_E \equiv \hbar\omega_E = 2JzS , \tag{2.11b}$$

$$\xi \equiv H_E (H_E + H_A)^{-1} , \tag{2.11c}$$

and, for a body-centered-cubic lattice which we pick for concreteness,

$$\gamma_{\vec{p}} = \cos\tfrac{1}{2}p_x \cos\tfrac{1}{2}p_y \cos\tfrac{1}{2}p_z . \tag{2.12}$$

(We use the abbreviations $a_{\vec{1}} = a_{\vec{k}_1}$, $b_{-\vec{2}} = b_{-\vec{k}_2}$, $\gamma_{\vec{1}-\vec{4}} = \gamma_{\vec{k}_1-\vec{k}_4}$, etc.) Throughout this paper we measure lengths in units of the spacing of each sublattice, i.e., wave numbers are dimensionless. Also, in Eq. (2.10c), the Kronecker δ, $\delta_{\vec{K}}(\vec{1}+\vec{2}-\vec{3}-\vec{4})$ expresses the conservation of momentum to within a reciprocal-lattice vector $\vec{K}$.

The Hamiltonian $\mathcal{H}_{DM}^{0\prime}$ may be diagonalized by the transformation[2,25]

$$a_{\vec{y}}^{\dagger}=l_{\vec{y}}\alpha_{\vec{y}}^{\dagger}+m_{\vec{y}}\beta_{-\vec{y}}\ , \tag{2.13a}$$

$$b_{-\vec{y}}=m_{\vec{y}}\alpha_{\vec{y}}^{\dagger}+l_{\vec{y}}\beta_{-\vec{y}}\ , \tag{2.13b}$$

where

$$l_{\vec{y}}=[(1+\epsilon_{\vec{y}})/2\epsilon_{\vec{y}}]^{1/2}\ , \tag{2.14a}$$

$$m_{\vec{y}}=-[(1-\epsilon_{\vec{y}})/2\epsilon_{\vec{y}}]^{1/2}\equiv -x_{\vec{y}}\,l_{\vec{y}}\ , \tag{2.14b}$$

with

$$\epsilon_{\vec{y}}=(1-\xi^2\gamma_{\vec{y}}^2)^{1/2}=E_{\vec{y}}(H_E+H_A)^{-1} \tag{2.15}$$

denoting a dimensionless energy. The Hamiltonian then becomes[61]

$$\mathcal{H}_{DM}=E_{DM}^0+\mathcal{H}_{DM}^0+V_{DM}\ , \tag{2.16}$$

where E_{DM}^0 is the ground-state energy in the spin-wave approximation,[3,4] and

$$\mathcal{H}_{DM}^0=(H_E+H_A)\sum_{\vec{y}}\epsilon_{\vec{y}}(\alpha_{\vec{y}}^{\dagger}\alpha_{\vec{y}}+\beta_{\vec{y}}^{\dagger}\beta_{\vec{y}})\ , \tag{2.17a}$$

$$\begin{aligned}V_{DM}=-\frac{H_E}{4NS}\sum_{\vec{1}\vec{2}\vec{3}\vec{4}}\delta_{\vec{K}}(\vec{1}+\vec{2}-\vec{3}-\vec{4})\,l_{\vec{1}}l_{\vec{2}}l_{\vec{3}}l_{\vec{4}}\times(&\alpha_{\vec{1}}^{\dagger}\alpha_{\vec{2}}^{\dagger}\alpha_{\vec{3}}\alpha_{\vec{4}}\Phi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}}+2\alpha_{\vec{1}}^{\dagger}\beta_{-\vec{2}}\alpha_{\vec{3}}\alpha_{\vec{4}}\Phi^{(2)}_{\vec{1}\vec{2}\vec{3}\vec{4}}+2\alpha_{\vec{1}}^{\dagger}\alpha_{\vec{2}}^{\dagger}\alpha_{\vec{3}}\beta_{-\vec{4}}^{\dagger}\Phi^{(3)}_{\vec{1}\vec{2}\vec{3}\vec{4}}\\&+4\alpha_{\vec{1}}^{\dagger}\beta_{-\vec{2}}\alpha_{\vec{3}}\beta_{-\vec{4}}^{\dagger}\Phi^{(4)}_{\vec{1}\vec{2}\vec{3}\vec{4}}+2\beta_{-\vec{1}}\beta_{-\vec{2}}\alpha_{\vec{3}}\beta_{-\vec{4}}^{\dagger}\Phi^{(5)}_{\vec{1}\vec{2}\vec{3}\vec{4}}+2\alpha_{\vec{1}}^{\dagger}\beta_{-\vec{2}}\beta_{-\vec{3}}^{\dagger}\beta_{-\vec{4}}^{\dagger}\Phi^{(6)}_{\vec{1}\vec{2}\vec{3}\vec{4}}\\&+\alpha_{\vec{1}}^{\dagger}\alpha_{\vec{2}}^{\dagger}\beta_{-\vec{3}}^{\dagger}\beta_{-\vec{4}}^{\dagger}\Phi^{(7)}_{\vec{1}\vec{2}\vec{3}\vec{4}}+\beta_{-\vec{1}}\beta_{-\vec{2}}\alpha_{\vec{3}}\alpha_{\vec{4}}\Phi^{(8)}_{\vec{1}\vec{2}\vec{3}\vec{4}}+\beta_{-\vec{1}}\beta_{-\vec{2}}\beta_{-\vec{3}}^{\dagger}\beta_{-\vec{4}}^{\dagger}\Phi^{(9)}_{\vec{1}\vec{2}\vec{3}\vec{4}})\ ,\end{aligned} \tag{2.17b}$$

where[62]

$$\begin{aligned}\Phi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}}=\Phi^{(9)}_{\vec{1}\vec{2}\vec{3}\vec{4}}=(&\gamma_{\vec{1}-\vec{4}}x_{\vec{1}}x_{\vec{4}}+\gamma_{\vec{1}-\vec{3}}x_{\vec{1}}x_{\vec{3}}+\gamma_{\vec{2}-\vec{4}}x_{\vec{2}}x_{\vec{4}}+\gamma_{\vec{2}-\vec{3}}x_{\vec{2}}x_{\vec{3}}\\&-\gamma_{\vec{1}}x_{\vec{2}}x_{\vec{3}}x_{\vec{4}}-\gamma_{\vec{2}}x_{\vec{1}}x_{\vec{3}}x_{\vec{4}}-\gamma_{\vec{2}}x_{\vec{2}}-\gamma_{\vec{1}}x_{\vec{1}}+2q_A+2q_A x_{\vec{1}}x_{\vec{2}}x_{\vec{3}}x_{\vec{4}})\ ,\end{aligned} \tag{2.18a}$$

$$\begin{aligned}\Phi^{(2)}_{\vec{1}\vec{2}\vec{3}\vec{4}}=\Phi^{(6)}_{\vec{2}\vec{1}\vec{3}\vec{4}}=(&-\gamma_{\vec{2}-\vec{4}}x_{\vec{4}}-\gamma_{\vec{2}-\vec{3}}x_{\vec{3}}-\gamma_{\vec{1}-\vec{4}}x_{\vec{1}}x_{\vec{2}}x_{\vec{4}}-\gamma_{\vec{1}-\vec{3}}x_{\vec{1}}x_{\vec{2}}x_{\vec{3}}\\&+\gamma_{\vec{1}}x_{\vec{3}}x_{\vec{4}}+\gamma_{\vec{2}}x_{\vec{1}}x_{\vec{2}}x_{\vec{3}}x_{\vec{4}}+\gamma_{\vec{2}}+\gamma_{\vec{1}}x_{\vec{1}}x_{\vec{2}}-2q_A x_{\vec{2}}-2q_A x_{\vec{1}}x_{\vec{3}}x_{\vec{4}})\ ,\end{aligned} \tag{2.18b}$$

$$\begin{aligned}\Phi^{(3)}_{\vec{1}\vec{2}\vec{3}\vec{4}}=\Phi^{(5)}_{\vec{1}\vec{2}\vec{4}\vec{3}}=(&-\gamma_{\vec{2}-\vec{4}}x_{\vec{2}}-\gamma_{\vec{1}-\vec{4}}x_{\vec{1}}-\gamma_{\vec{2}-\vec{4}}x_{\vec{1}}x_{\vec{3}}x_{\vec{4}}-\gamma_{\vec{2}-\vec{3}}x_{\vec{2}}x_{\vec{3}}x_{\vec{4}}\\&+\gamma_{\vec{1}}x_{\vec{2}}x_{\vec{3}}+\gamma_{\vec{2}}x_{\vec{1}}x_{\vec{3}}+\gamma_{\vec{2}}x_{\vec{2}}x_{\vec{4}}+\gamma_{\vec{1}}x_{\vec{1}}x_{\vec{4}}-2q_A x_{\vec{4}}-2q_A x_{\vec{1}}x_{\vec{2}}x_{\vec{3}})\ ,\end{aligned} \tag{2.18c}$$

$$\Phi^{(4)}_{\vec{1}\vec{2}\vec{3}\vec{4}}=(\gamma_{\vec{2}-\vec{4}}+\gamma_{\vec{1}-\vec{4}}x_{\vec{1}}x_{\vec{2}}+\gamma_{\vec{1}-\vec{4}}x_{\vec{3}}x_{\vec{4}}+\gamma_{\vec{1}-\vec{3}}x_{\vec{1}}x_{\vec{2}}x_{\vec{3}}x_{\vec{4}}-\gamma_{\vec{1}}x_{\vec{3}}-\gamma_{\vec{2}}x_{\vec{1}}x_{\vec{2}}x_{\vec{3}}-\gamma_{\vec{2}}x_{\vec{4}}-\gamma_{\vec{1}}x_{\vec{1}}x_{\vec{2}}x_{\vec{4}}+2q_A x_{\vec{1}}x_{\vec{3}}+2q_A x_{\vec{2}}x_{\vec{4}})\ , \tag{2.18d}$$

$$\begin{aligned}\Phi^{(7)}_{\vec{1}\vec{2}\vec{3}\vec{4}}=\Phi^{(8)}_{\vec{1}\vec{2}\vec{3}\vec{4}}=(&\gamma_{\vec{2}-\vec{4}}x_{\vec{2}}x_{\vec{3}}+\gamma_{\vec{2}-\vec{3}}x_{\vec{2}}x_{\vec{4}}+\gamma_{\vec{2}-\vec{3}}x_{\vec{1}}x_{\vec{3}}+\gamma_{\vec{2}-\vec{4}}x_{\vec{1}}x_{\vec{4}}\\&-\gamma_{\vec{1}}x_{\vec{1}}x_{\vec{3}}x_{\vec{4}}-\gamma_{\vec{2}}x_{\vec{2}}x_{\vec{3}}x_{\vec{4}}-\gamma_{\vec{1}}x_{\vec{2}}-\gamma_{\vec{2}}x_{\vec{1}}+2q_A x_{\vec{3}}x_{\vec{4}}+2q_A x_{\vec{1}}x_{\vec{2}})\ ,\end{aligned} \tag{2.18e}$$

with

$$q_A=2DS/H_E\ . \tag{2.19}$$

Given the boson Hamiltonian in Eq. (2.16), we may introduce thermodynamic Green's functions in the usual way.[63] Because of the presence of the two sublattices, it is convenient to work in terms of matrix Green's functions, analogous to the Nambu[64] matrices for superconductors, or similar functions for the Bose liquid.[26] For imaginary times, we have

$$G_{\alpha\alpha}(\vec{k},t)=-i\langle T(\alpha_{\vec{k}}(t)\,\alpha_{\vec{k}}^{\dagger}(0))\rangle\ , \tag{2.20a}$$

$$G_{\alpha\beta}(\vec{k},t)=-i\langle T(\alpha_{\vec{k}}(t)\,\beta_{-\vec{k}}(0))\rangle\ , \tag{2.20b}$$

$$G_{\beta\alpha}(\vec{k},t)=-i\langle T(\beta_{-\vec{k}}^{\dagger}(t)\,\alpha_{\vec{k}}^{\dagger}(0))\rangle\ , \tag{2.20c}$$

$$G_{\beta\beta}(\vec{k},t)=-i\langle T(\beta_{-\vec{k}}^{\dagger}(t)\,\beta_{-\vec{k}}(0))\rangle\ , \tag{2.20d}$$

where the angular bracket denotes an average over an ensemble at a temperature $(\beta k_B)^{-1}$, and here T

is the time-ordering operator. The frequency-dependent functions are defined by Fourier transforms[65]

$$G_{\mu\nu}(\vec{k}, z) = \hbar^{-1}\int_0^{-i\beta\hbar} G_{\mu\nu}(\vec{k}, t)\, e^{izt}\, dt \qquad (2.21)$$

for $z = 2\pi n i(\beta\hbar)^{-1}$, with n an integer. For other values of z, $G_{\mu\nu}(\vec{k}, z)$ is defined by the usual analytic continuation procedure.[66] The Green's function satisfies a matrix Dyson equation:

$$G_{\mu\nu}(\vec{k}, z) = G^0_{\mu\nu}(\vec{k}, z) + \sum_{\gamma\delta} G^0_{\mu\gamma}(\vec{k}, z)\Sigma_{\gamma\delta}(\vec{k}, z)\, G_{\delta\nu}(\vec{k}, z)\,, \qquad (2.22)$$

where the Greek-letter subscripts range over the values α and β. The unperturbed Green's functions $G^0_{\mu\nu}(\mathrm{k}, z)$ are given by

$$G^0_{\alpha\alpha}(\vec{k}, z) = (\hbar z - E_{\vec{k}})^{-1}\,, \qquad (2.23a)$$

$$G^0_{\alpha\beta}(\vec{k}, z) = G^0_{\beta\alpha}(\vec{k}, z) = 0\,, \qquad (2.23b)$$

$$G^0_{\beta\beta}(\vec{k}, z) = (-\hbar z - E_{\vec{k}})^{-1}\,, \qquad (2.23c)$$

where $E_{\vec{k}}$ is defined in Eq. (2.15). The self-energy $\Sigma_{\mu\nu}(\vec{k}, z)$ can be represented by the usual kind of diagrammatic expansion; we shall use the "occupation number" formulation.[53,54] The fact that the perturbing Hamiltonian is non-Hermitian is of no consequence for this formalism. From the symmetry of the Hamiltonian with respect to the interchange $\alpha_{\vec{3}} \rightleftharpoons \beta^\dagger_{-\vec{3}}$ we have the following relations:

$$\Sigma_{\alpha\alpha}(\vec{k}, z) = \Sigma_{\beta\beta}(-\vec{k}, -z), \qquad (2.24a)$$

$$\Sigma_{\alpha\beta}(\vec{k}, z) = \Sigma_{\beta\alpha}(-\vec{k}, -z). \qquad (2.24b)$$

In addition, it is clear from the diagrammatic expansion of Baym and Sessler,[53] for instance, that

$$\Sigma_{\mu\nu}(\vec{k}, z^*)^* = \Sigma_{\mu\nu}(\vec{k}, z)\,, \qquad (2.25)$$

because the potential V_{DM} in Eq. (2.16) is real (once again this result does not depend on Hermiticity). Moreover, $\Sigma_{\mu\nu}(\vec{k}, z)$ is even in $\vec{k}$ for a cubic lattice. Henceforth, we shall refer to Eqs. (2.24) and (2.25) as the general symmetry relations.

Given expressions for the elements of the matrix $\Sigma_{\mu\nu}(\vec{k}, z)$, we may calculate the Green's functions $G_{\mu\nu}(\vec{k}, z)$ by using Eq. (2.22). As will be shown in what follows, $\Sigma_{\mu\nu}(\vec{k}, z)$ is small compared to the unperturbed energy $E_{\vec{k}}$, so that, to leading order in $||\underline{\Sigma}(\vec{k}, z)||/E_{\vec{k}}$, the inversion of Eq. (2.22) may be written as

$$G_{\alpha\alpha}(\vec{k}, z) = [\hbar z - E_{\vec{k}} - \Sigma_{\alpha\alpha}(\vec{k}, z)]^{-1}\,, \qquad (2.26a)$$

$$G_{\beta\beta}(\vec{k}, z) = [-\hbar z - E_{\vec{k}} - \Sigma_{\beta\beta}(\vec{k}, z)]^{-1}\,, \qquad (2.26b)$$

$$G_{\alpha\beta}(\vec{k}, z) = -\Sigma_{\alpha\beta}(\vec{k}, z)[\hbar z - E_{\vec{k}} - \Sigma_{\alpha\alpha}(\mathrm{k}, z)]^{-1} \times [\hbar z + E_{\vec{k}} + \Sigma_{\beta\beta}(\vec{k}, z)]^{-1}\,, \qquad (2.26c)$$

$$G_{\beta\alpha}(\vec{k}, z) = -\Sigma_{\beta\alpha}(\vec{k}, z)[\hbar z - E_{\vec{k}} - \Sigma_{\alpha\alpha}(\vec{k}, z)]^{-1} \times [\hbar z + E_{\vec{k}} + \Sigma_{\beta\beta}(\vec{k}, z)]^{-1}\,. \qquad (2.26d)$$

(A more complete version of this analysis is presented in Appendix I.) Thus the damping of the α magnons is given by[67]

$$\Sigma''_{\alpha\alpha}(\vec{k}, \omega) \equiv \mathrm{Im}\Sigma_{\alpha\alpha}(\vec{k}, \omega - i0^+) \equiv \hbar\Gamma(\vec{k}, \omega)\,, \qquad (2.27)$$

and to lowest order in $||\underline{\Sigma}||/E_{\vec{k}}$ it does not involve the other elements of the $\underline{\Sigma}$ matrix.

C. Spin-Correlation Functions

Let us introduce Fourier transforms of the spin operators in Eq. (2.2):

$$S^+_a(\vec{k}) = (2S)^{1/2}[a_{\vec{k}} + A_{\vec{k}}]\,, \qquad (2.28a)$$

$$S^-_a(\vec{k}) = (2S)^{1/2}[a^\dagger_{\vec{k}}]\,, \qquad (2.28b)$$

$$S^+_b(\vec{k}) = (2S)^{1/2}[b^\dagger_{-\vec{k}} + B^\dagger_{-\vec{k}}]\,, \qquad (2.28c)$$

$$S^-_b(\vec{k}) = (2S)^{1/2}[b_{-\vec{k}}]\,, \qquad (2.28d)$$

where

$$-2SA_{\vec{k}} = N^{-1}\sum_{\vec{2}\vec{3}\vec{4}} \delta_{\vec{K}}(\vec{k} + \vec{2} - \vec{3} - \vec{4})\, a^\dagger_{\vec{2}} a_{\vec{3}} a_{\vec{4}}\,, \qquad (2.29a)$$

$$-2SB^\dagger_{-\vec{k}} = N^{-1}\sum_{\vec{2}\vec{3}\vec{4}} \delta_{\vec{K}}(\vec{k} + \vec{2} - \vec{3} - \vec{4})\, b^\dagger_{-\vec{3}} b^\dagger_{-\vec{4}} b_{-\vec{2}}\,. \qquad (2.29b)$$

Using these operators, we define a Green's-function matrix

$$2S\mathcal{G}_{mn}(\vec{k}, t) \equiv -i\langle T(S^+_m(\vec{k}, t) S^-_n(\vec{k}, t))\rangle, \qquad m, n = a, b \qquad (2.30)$$

and a corresponding matrix related to the α, β operators, which we write in terms of the Fourier transform of Eq. (2.30) as[68]

$$\mathcal{G}_{\mu\nu}(\vec{k}, z) \equiv \sum_{m,n} (U^{-1}_{\vec{k}})_{\mu m}\, \mathcal{G}_{mn}(\vec{k}, z)\, (U^{-1}_{\vec{k}})_{n\nu}\,, \qquad (2.31)$$

where $\underline{U}^{-1}_{\vec{k}}$ is the matrix for the transformation, Eq. (2.13):

$$\underline{U}^{-1}_{\vec{k}} = \begin{pmatrix} l_{\vec{k}} & -m_{\vec{k}} \\ -m_{\vec{k}} & l_{\vec{k}} \end{pmatrix}\,. \qquad (2.32)$$

As shown in Appendix C, the matrix $\mathcal{G}_{\mu\nu}$ is related to $G_{\mu\nu}$ by the equation

$$\mathcal{G}_{\mu\nu} = \sum_\rho (\delta_{\mu\rho} + \Lambda_{\mu\rho})\, G_{\rho\nu}\,, \qquad (2.33)$$

where the matrix $\Lambda_{\mu\nu}$ has a diagrammatic expansion (see Appendix C) similar to $\Sigma_{\mu\nu}$, and $\delta_{\mu\nu}$ is the unit matrix. Given the matrix $\mathcal{G}_{\mu\nu}$, we can obtain the matrix $\mathcal{G}_{mn}$ by inverting Eq. (2.31), and any desired spin Green's function, by taking appropriate linear combinations of its elements. Of particular interest are the correlation functions for the "staggered" and "total" spins which are defined, respectively, by

$$Q_{\vec{k}}^{+} = S_a^{+}(\vec{k}) - S_b^{+}(\vec{k}) , \tag{2.34a}$$

$$S_{\vec{k}}^{+} = S_a^{+}(\vec{k}) + S_b^{+}(\vec{k}) . \tag{2.34b}$$

The Fourier transform of the staggered-spin Green's function is given in terms of the $\mathcal{G}_{mn}$ as

$$\mathcal{G}_Q^{+-}(\vec{k}, z) = 2S[\mathcal{G}_{aa}(\vec{k}, z) + \mathcal{G}_{bb}(\vec{k}, z) - \mathcal{G}_{ab}(\vec{k}, z) - \mathcal{G}_{ba}(\vec{k}, z)] , \tag{2.35}$$

and, similarly, for $\mathcal{G}_S^{+-}(\vec{k}, z)$. From the properties of the transformation $\underline{U}_{\vec{k}}$ [Eqs. (2.32) and (2.14)], we can easily show that

$$\mathcal{G}_Q^{+-}(\vec{k}, z) = 2S(l_{\vec{k}} - m_{\vec{k}})^2 [\mathcal{G}_{\alpha\alpha}(\vec{k}, z) + \mathcal{G}_{\beta\beta}(\vec{k}, z) - \mathcal{G}_{\alpha\beta}(\vec{k}, z) - \mathcal{G}_{\beta\alpha}(\vec{k}, z)] , \tag{2.36a}$$

and, similarly,

$$\mathcal{G}_S^{+-}(\vec{k}, z) = 2S(l_{\vec{k}} + m_{\vec{k}})^2 [\mathcal{G}_{\alpha\alpha}(\vec{k}, z) + \mathcal{G}_{\beta\beta}(\vec{k}, z) + \mathcal{G}_{\alpha\beta}(\vec{k}, z) + \mathcal{G}_{\beta\alpha}(\vec{k}, z)] . \tag{2.36b}$$

If we neglect spin-wave interactions, we have $\mathcal{G}_{\mu\nu} = G_{\mu\nu} = G_{\mu\nu}^0$, so that

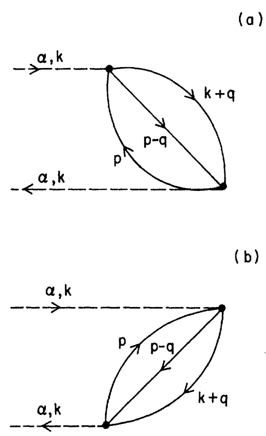

FIG. 1. Lowest- (second-) order diagrams which contribute to the decay rate of antiferromagnetic magnons on resonance. The internal lines can either be all α magnons or one α magnon and two β magnons. These choices correspond, respectively, to the first, and the sum of the last two terms in the bracket on the right-hand side of Eq. (2.39). Our convention is that lines going from right to left are hole lines. Interaction vertices are represented by dots. The dashed lines connected to the external vertices represent the propagators which must be attached to the mass operator, in order to form a Green's function. These propagators are, of course, not included in the evaluation of the mass operator.

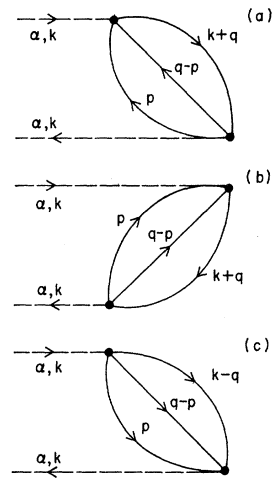

FIG. 2. Additional lowest-order diagrams which contribute to the decay rate off-resonance. These scattering processes cannot conserve energy and momentum on-resonance because of the shape of the magnon dispersion relation (see Appendix E). The contribution of (c) to the damping is negligible even off resonance, as long as $\vec{k}$ and ω are small.

$$\mathcal{G}_Q^{+-}(\vec{k}, z) = \frac{4S}{\epsilon_{\vec{k}}} \left(\frac{1}{\hbar z - H_E \epsilon_{\vec{k}}} - \frac{1}{\hbar z + H_E \epsilon_{\vec{k}}} \right) , \tag{2.37a}$$

$$\mathcal{G}_S^{+-}(\vec{k}, z) = S\epsilon_{\vec{k}} \left(\frac{1}{\hbar z - H_E \epsilon_{\vec{k}}} - \frac{1}{\hbar z + H_E \epsilon_{\vec{k}}} \right) . \tag{2.37b}$$

In writing Eq. (2.37) we have kept only the leading term in $\epsilon_{\vec{k}}$.

D. Perturbation Expansion for $\Sigma_{\alpha\alpha}$

Compared to the term $\mathcal{H}_{\rm DM}^0$, the interaction term $V_{\rm DM}$ in the Hamiltonian (2.16) contains an extra pair of operators α or β, extra momentum sums, and a factor of S^{-1}. Roughly speaking, at low temperatures each pair of operators yields either an occupation number n proportional to T^3 or a factor $1+n$, and the momentum sums yield cutoffs and factors of $1/z$, where z is the number of nearest neighbors (or the third power of the interaction range, for long-range interactions). Thus an expansion in powers of V_{DM} will be an expansion in powers of the temperature, of z^{-1}, and, for zero anisotropy, of S^{-1}. We shall consider this expansion in detail in Sec. V; for the moment, we calculate the contributions to $\tilde{\Sigma}'_{\alpha\alpha}(\vec{k}, z)$ from the lowest-order diagrams, namely, those with two vertices,[69] which are shown in Figs. 1 and 2. Note that, in lowest order, a diagram gives a nonzero contribution only if energy and momentum are conserved in the intermediate state. We shall initially consider the decay rate for a magnon which is "on-resonance" ($\hbar\omega = E_{\vec{k}}$), i.e., we calculate $\Gamma_{\vec{k}}$

$\equiv \Gamma(\vec{k}, \hbar^{-1}E_{\vec{k}})$ [see Eq. (2.27)]. As shown in Appendix E, the diagrams of Fig. 2 cannot conserve momentum and energy in this case, and hence give no contribution. We are left with those of Fig. 1 which, according to the rules of Refs. 53 and 54, give

$$\Sigma''_{\alpha\alpha}(\vec{k}, \hbar^{-1}E_{\vec{k}}) \equiv \hbar\Gamma_{\vec{k}} = \pi(H_E/4NS)^2(1-e^{-\beta E_{\vec{k}}}) \times \sum_{\vec{p},\vec{s}} n_{\vec{p}}(1+n_{\vec{s}})(1+n_{\vec{r}})\,\delta[H_E(\epsilon_{\vec{k}}+\epsilon_{\vec{p}}-\epsilon_{\vec{r}}-\epsilon_{\vec{s}})] \times M_{22}(\vec{k},\vec{p},\vec{s},\vec{r}) , \tag{2.38}$$

where the "matrix element" M_{22} is given by[70]

$$M_{22}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) = 8 l_{\vec{1}}^2 l_{\vec{2}}^2 l_{\vec{3}}^2 l_{\vec{4}}^2 (\Phi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}}\Phi^{(1)}_{\vec{3}\vec{4}\vec{1}\vec{2}} + \Phi^{(4)}_{\vec{1},-\vec{4},\vec{3},-\vec{2}}\Phi^{(4)}_{\vec{3},-\vec{2},\vec{1},-\vec{4}} + \Phi^{(4)}_{\vec{1},-\vec{3},\vec{4},-\vec{2}}\Phi^{(4)}_{\vec{4},-\vec{2},\vec{1},-\vec{3}}) , \tag{2.39}$$

and the occupation number is

$$n_{\vec{p}} \equiv n(\epsilon_{\vec{p}}) = (e^{\beta E_{\vec{p}}}-1)^{-1} \equiv (e^{2\epsilon_{\vec{p}}/\tau}-1)^{-1} . \tag{2.40}$$

In (2.38) and below, momentum conservation is implied, so that $\vec{k}_1+\vec{k}_2=\vec{k}_3+\vec{k}_4$ and $\vec{k}+\vec{p}=\vec{r}+\vec{s}$. In the derivation of Eq. (2.38) we have combined diagrams obtained from one another by interchanging the "time order," which yields the factor $(1-e^{-\beta E_{\vec{k}}})$ because of the relation

$$n_{\vec{2}}(1+n_{\vec{3}})(1+n_{\vec{4}}) - (1+n_{\vec{2}})n_{\vec{3}}n_{\vec{4}} = n_{\vec{2}}(1+n_{\vec{3}})(1+n_{\vec{4}})[1+n(\epsilon_{\vec{3}}+\epsilon_{\vec{4}}-\epsilon_{\vec{2}})]^{-1} . \tag{2.41}$$

The last equality in (2.40) defines the dimensionless temperature used in this paper:

$$\tau \equiv k_BT/JzS = 2/\beta H_E . \tag{2.42}$$

III. DAMPING OF ON-RESONANCE MAGNONS IN QUANTUM LOW-TEMPERATURE LIMIT OF ISOTROPIC MODEL IN LOWEST BORN APPROXIMATION

We shall carry out the evaluation of the damping constant $\Gamma_{\vec{k}}$ within the lowest Born approximation as given by Eq. (2.38) in the long-wavelength limit $k \ll 1$. In this section, we consider only magnons on resonance ($\hbar\omega = E_{\vec{k}}$) and neglect the damping of intermediate states. Both of these restrictions will be lifted in Sec. IV. Here we treat the quantum low-temperature limit $k_BT \ll JzS$, i.e., $\tau \ll 1$. The classical low-temperature limit $S\to\infty$, $T\ll T_N$, $JzS \ll k_BT \ll JzS^2$, i.e., $1\ll\tau\ll S$, will be treated in Sec. VI. Since the long-wavelength quantum low-temperature limit is characterized by two small parameters, $\epsilon_{\vec{k}}$ and τ, we must recognize the possible existence of various regimes depending on the relative size of these small parameters. Accordingly, in Sec. III A we give a discussion of the possible shapes of the scattering surface determined by the energy and momentum conservation laws for the processes in question. The existence of scattering surfaces of qualitatively different shapes gives rise to different regimes for $\Gamma_{\vec{k}}$. In Secs. III B–III E, we present the detailed evaluation of $\Gamma_{\vec{k}}$ to lowest order in both k and τ within the various regimes described in Sec. III A.

A. Long-Wavelength Regimes

It is clear that we shall want to use the long-wavelength approximations for the matrix element M_{22} in Eq. (2.39). At long wavelength we also have, from Eq. (2.15),

$$\epsilon_{\vec{k}} \approx \tfrac{1}{2}k . \tag{3.1}$$

This approximation can be used for the energies in the occupation numbers [Eq. (2.41)]. However, if we replace the energies inside the δ function by their linear approximations, we will not, in general, describe the scattering surface correctly. By the scattering surface we mean that surface in $\vec{q}$ space, which for fixed values of $\vec{p}$ and $\vec{k}$ is determined by

$$\epsilon_{\vec{k}}+\epsilon_{\vec{p}}-\epsilon_{\vec{p}-\vec{q}}-\epsilon_{\vec{k}+\vec{q}} = 0 . \tag{3.2}$$

First of all, note that the point $\vec{q}=\frac{1}{2}(\vec{p}-\vec{k})$ is a center of inversion symmetry, i.e., if $\vec{q}$ satisfies Eq. (3.2), then so does $\vec{q}\,'=\vec{p}-\vec{k}-\vec{q}$. Next, let us examine the detailed shape of the scattering surface. Using the linear approximation for the energies, we write Eq. (3.2) as

$$k+p-|\vec{k}+\vec{q}|-|\vec{p}-\vec{q}| = 0 , \tag{3.3}$$

which is the equation of an ellipsoid centered about the point $\vec{q}=\frac{1}{2}(\vec{p}-\vec{k})$. This description of the scattering surface will not be valid when either k is much larger than p, or vice versa.

To see this quantitatively, let us study the function $F(\vec{q})$, where

$$F(\vec{q}) = \epsilon_{\vec{p}-\vec{q}}+\epsilon_{\vec{k}+\vec{q}}-\epsilon_{\vec{p}}-\epsilon_{\vec{k}} . \tag{3.4}$$

In particular, if we set

$$\vec{q}=\vec{q}(t)=\tfrac{1}{2}(\vec{p}-\vec{k})+t\vec{n} , \tag{3.5}$$

where $\vec{n}$ is a unit vector perpendicular to $\vec{p}+\vec{k}$, then $F(\vec{q}(t))$ is a monotonically increasing function of $|t|$. Thus if $F(\vec{q})$ is positive for $t=0$, there can be no solution of $F(\vec{q})=0$ in the plane spanned by $\vec{q}(t)$ for various t and $\vec{n}$. Accordingly, the condition that the scattering surface consist of two disjoint pieces is

$$2\epsilon_{(\vec{k}+\vec{p})/2} \geq \epsilon_{\vec{p}}+\epsilon_{\vec{k}} . \tag{3.6}$$

In order to manifest this condition, we need to keep the cubic terms in the expansion for $\epsilon_{\vec{k}}$. From Eq. (2.15) we find

$$2\epsilon_{\vec{k}} = k[1-k^2g(\hat{k})] , \tag{3.7}$$

where

$$48g(\hat{k}) = 3-k^{-4}(k_x^4+k_y^4+k_z^4) . \tag{3.8}$$

For convenience, we shall generally approximate $g(\hat{k})$ by its average over angles, which we denote by g_{av}:

$$g_{av} = \tfrac{1}{20} \ . \tag{3.9}$$

Using these expansions, we write the condition equation (3.6) as

$$|\vec{k}+\vec{p}| - \tfrac{1}{4} g_{av} |\vec{k}+\vec{p}|^3 > k + p - g_{av}(k^3+p^3) \ . \tag{3.10}$$

There are now three cases to consider: (i) $k^3 \geq p$, (ii) $k^3 \leq p \leq k^{1/3}$, and (iii) $k \leq p^3$. The cubic terms in Eq. (3.10) are only important when k^3 (or p^3) is comparable to or larger than p (or k). In other words, the cubic terms are important in cases (i) and (iii), and then we write the condition for disjoint surfaces, Eq. (3.10), as

$$\cos\theta_{\vec{k}\vec{p}} > 1 - 3g_{av} k^3/4p, \quad k^3 \geq p \tag{3.11a}$$

$$\cos\theta_{\vec{k}\vec{p}} > 1 - 3g_{av} p^3/4k, \quad p^3 \geq k \tag{3.11b}$$

where $\cos\theta_{\vec{k}\vec{p}} = \hat{k}\cdot\hat{p}$. In case (ii), it is clear that disjoint surfaces exist only for an infinitesimal range of angles $\theta_{\vec{k}\vec{p}}$. In case (ii) we can therefore approximate the scattering surface by an ellipsoid using Eq. (3.3).

On the other hand, for cases (i) and (iii), it is obvious that Eq. (3.3) no longer describes the scattering surface adequately, since in these cases the scattering surface consists of two disjoint surfaces. To describe these disjoint surfaces, for instance, in case (iii) when $p^3 \geq k$, we expand Eq. (3.2) assuming p to be the dominant momentum. Then we obtain the equation of the scattering surface near $q=0$ as

$$k - |\vec{k}+\vec{q}| + \vec{q}\cdot\vec{v}_{\vec{p}} = 0 \ , \tag{3.12a}$$

where $\vec{v}_{\vec{p}} \equiv 2\vec{\nabla}_{\vec{p}}\epsilon_{\vec{p}}$. (We use this convention, because then $v_p = 1$ for small p.) Equation (3.12a) may be solved exactly for q:

$$q = 2k\left(\frac{\hat{q}\cdot\vec{v}_{\vec{p}} - \hat{q}\cdot\hat{k}}{1-(\hat{q}\cdot\vec{v}_{\vec{p}})^2}\right) \ . \tag{3.12b}$$

From this result we see that q is of order k except when $(\hat{q}\cdot\vec{v}_{\vec{p}})^2$ approaches unity, i.e., except when $\cos^2\theta_{\vec{q}\vec{p}} \equiv (\hat{q}\cdot\hat{p})^2 \sim 1$, in which case it can become as large as $\sim k/p^2$. To describe the regime $(\hat{q}\cdot\hat{p})^2 \sim 1$ we may use the approximation

$$\hat{q}\cdot\vec{v}_{\vec{p}} = \hat{q}\cdot\hat{p}\,[1 - 3p^2 g(\hat{p})] \ , \tag{3.13}$$

which can be obtained from Eq. (3.7). This approximation breaks down as $(\hat{q}\cdot\hat{p})^2$ deviates from unity. In that case, however, the curvature has relatively little effect on the denominator in Eq. (3.12b). Hence Eqs. (3.12) and (3.13) provide a good description of the scattering surface when $p^3 > k$. To illustrate this discussion, we show in Fig. 3 the scattering surface for three sets of values of the momenta, calculated using the exact equation (3.2). From these diagrams or from Eq. (3.10), one sees that disjoint surfaces occur when $k < cp^3$, where c is a sufficiently small quantity ($c \sim g_{av}$) so that this regime is a very extreme case.

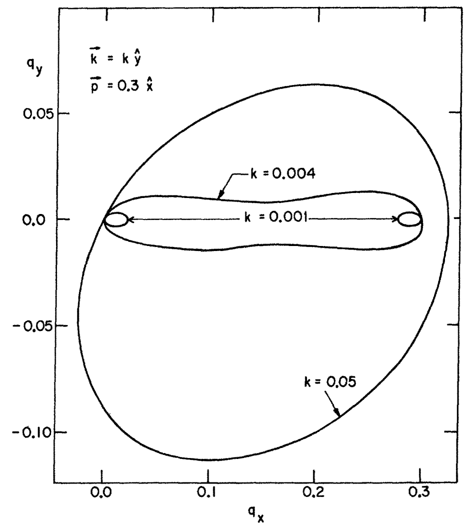

FIG. 3. Scattering surface for fixed incoming momenta $\vec{k}$ and $\vec{p}$ defined as a function of $\vec{q}$ by $\epsilon_{\vec{k}} + \epsilon_{\vec{p}} = \epsilon_{\vec{k}+\vec{q}} + \epsilon_{\vec{p}-\vec{q}}$. Here we show a cross section of this surface for $q_z = 0$ for some cases when $\vec{k}$ and $\vec{p}$ lie in the x-y plane. The outermost and innermost curves are given to very good accuracy by Eqs. (3.3) and (3.12), respectively. For the middle curve $k \sim gp^3$, so that the scattering surface is in a transitional regime intermediate between the other two cases.

Let us now see how the shape of the scattering surface influences the evaluation of the decay rate in lowest Born approximation. From Eq. (2.38), it appears (and we shall see definitely later) that the dominant contributions to $\Gamma_{\vec{k}}$ come when $p \approx \tau$, i.e., when $\vec{p}$ is a thermal momentum. Thus in regime A,

$$\text{A: } \epsilon_{\vec{k}} \ll \tau^3 \ll 1 \ , \tag{3.14a}$$

we have essentially $k \ll p^3$, so that the scattering surface consists of two disjoint surfaces. In regime B,

$$\text{B: } \tau^3 \ll \epsilon_{\vec{k}} \ll \tau \ll 1 \ , \tag{3.14b}$$

we have a single scattering surface, but the temperature is still large enough ($\tau \gg \epsilon_{\vec{k}}$), so that reverse processes cannot be neglected, i.e., $e^{-\beta H_E \epsilon_{\vec{k}}}$ is of order unity. Regime C,

$$\text{C: } \tau \ll \epsilon_{\vec{k}} \ll \tau^{1/3} \ll 1 \ , \tag{3.14c}$$

is similar to B, except that here the temperature is low enough, so that reverse processes can be neglected. Finally, in regime D,

$$D\text{: } \tau^{1/3} \ll \epsilon_{\vec{k}} \ll 1 \ , \tag{3.14d}$$

we again have disjoint scattering surfaces for thermal momenta because for these $k^3 \gg p$.

B. Regime A

We now proceed to evaluate the lowest Born approximation for the decay rate $\Gamma_{\vec{k}}$ of resonant spin waves in regime A, where $\epsilon_{\vec{k}} \ll \tau^3$. For the moment, let us assume that the dominant contributions to $\Gamma_{\vec{k}}$ come from thermal values of p, i.e., $p \approx \tau$. For such values of p the scattering surface consists of two disjoint surfaces, one corresponding to values of $\vec{q}$ near $q=0$ and the other to values of q near $\vec{q}=\vec{p}-\vec{k}$. Since the surface near $\vec{q}=\vec{p}-\vec{k}$ can be obtained from the one near $\vec{q}=0$ by replacing $\vec{q}$ by $\vec{p}-\vec{k}-\vec{q}$, and since the integrand is symmetric in $\vec{k}+\vec{q}$ and $\vec{p}-\vec{q}$, both disjoint surfaces give identical contributions to $\Gamma_{\vec{k}}$. Taking twice the contribution of the surface near $q=0$ and using Eq. (3.12) to describe the scattering surface, we write Eq. (2.38) as

$$\Gamma_{\vec{k}} = \tfrac{1}{4}\pi\omega_E S^{-2}(2\pi)^{-6}(1-e^{-\beta H_E \epsilon_{\vec{k}}})\int d\vec{p}\int d\vec{q}$$
$$\times n_{\vec{s}}(1+n_{\vec{r}})(1+n_{\vec{s}})$$
$$\times M_{22}(\vec{k},\vec{p},\vec{r},\vec{s})\,\delta(2\epsilon_{\vec{k}}-2\epsilon_{\vec{s}}-2\vec{q}\cdot\vec{v}_{\vec{p}}) , \quad (3.15)$$

where $\vec{s}=\vec{k}+\vec{q}$ and $\vec{r}=\vec{p}-\vec{q}$. At long wavelengths we use the approximations of Eq. (A20) for the Φ coefficients, from which Eq. (2.39) becomes

$$\tfrac{1}{2}M_{22}(\vec{k},\vec{p},\vec{r},\vec{s}) = [(\hat{k}\cdot\hat{p}-1)(\hat{r}\cdot\hat{s}-1)$$
$$+(\hat{k}\cdot\hat{r}-1)(\hat{p}\cdot\hat{s}-1)$$
$$+(\hat{k}\cdot\hat{s}-1)(\hat{p}\cdot\hat{r}-1)]. \quad (3.16)$$

Before continuing with the detailed evaluation of $\Gamma_{\vec{k}}$, let us make some general comments on the form of Eq. (3.15). If we were to set $\vec{v}_{\vec{p}}=1$ in Eq. (3.15), then a dimensional argument would tell us that $\Gamma_{\vec{k}}$ is of the form

$$\Gamma_{\vec{k}} \sim \tau^5 f(k/\tau) . \quad (3.17)$$

In fact, however, the integrals thus obtained do not converge without the introduction of an additional cutoff. In regime A, this extra cutoff is provided by the "curvature energy" of the thermal magnons, as reflected by the fact that $|1-\vec{v}_{\vec{p}}|$ is of order p^2.

Since both k and q are small in comparison to p, we retain only the leading order in k/p and q/p. For this purpose we transform Eq. (3.16) into a more convenient form. Note that the energy δ function restricts the momenta $\vec{p}$, $\vec{q}$, and $\vec{k}$ to be on the scattering surface. In that case, M_{22} may be rewritten, using Eq. (A23), in the form

$$M_{22}(\vec{k},\vec{p},\vec{r},\vec{s}) = 2(\epsilon_{\vec{k}}\epsilon_{\vec{p}}\epsilon_{\vec{r}}\epsilon_{\vec{s}})^{-1}[\epsilon_{\vec{k}}^2\epsilon_{\vec{p}}^2(\hat{k}\cdot\hat{p}-1)^2$$
$$+\epsilon_{\vec{k}}^2\epsilon_{\vec{s}}^2(\hat{k}\cdot\hat{s}-1)^2+\epsilon_{\vec{k}}^2\epsilon_{\vec{r}}^2(\hat{k}\cdot\hat{r}-1)^2] , \quad (3.18)$$

since $\Delta\epsilon_{12}=0$. Thus, neglecting q and k in comparison to p, we find

$$M_{22}(\vec{k},\vec{p},\vec{r},\vec{s}) = (4\epsilon_{\vec{k}}/\epsilon_{\vec{p}})(\hat{k}\cdot\hat{p}-1)^2 . \quad (3.19)$$

In regime A, we may set $(1-e^{-\beta E_{\vec{k}}}) \approx \beta E_{\vec{k}}$ and $1+n_{\vec{s}} \approx (\beta E_{\vec{s}})^{-1}$. Transforming the integration variable from $\vec{q}$ to $\vec{s}\equiv\vec{k}+\vec{q}$ we have

$$\Gamma_{\vec{k}} = 2\omega_E S^{-2}(2\pi)^{-3}\epsilon_{\vec{k}}^2 I_A , \quad (3.20)$$

where

$$I_A = (2\pi)^{-2}\int d\vec{p}\int d\vec{s}\, n_{\vec{p}}(1+n_{\vec{p}})\, s^{-2}(\hat{k}\cdot\hat{v}_{\vec{p}}-1)^2$$
$$\times\delta(k-s+\vec{v}_{\vec{p}}\cdot(\vec{s}-\vec{k})) . \quad (3.21)$$

We choose a coordinate system for the $\vec{s}$ integral in which $\vec{v}_{\vec{p}}$ lies along the z axis, and we set

$$\hat{v}_{\vec{p}}\cdot\hat{s} = \nu , \quad (3.22a)$$

$$\hat{v}_{\vec{p}}\cdot\hat{k} = \mu . \quad (3.22b)$$

Then I_A becomes

$$I_A = (2\pi)^{-1}\int d\vec{p}\int_0^\infty ds\int_{-1}^1 d\nu\, n_{\vec{p}}(1+n_{\vec{p}})(\hat{k}\cdot\hat{v}_{\vec{p}}-1)^2$$
$$\times\delta(k-s+v_{\vec{p}}(s\nu-k\mu)) . \quad (3.23)$$

The ν integration satisfies the δ-function condition providing $|\nu_0|\le 1$, where

$$\nu_0 = [(\nu_{\vec{p}}\mu-1)k+s]/v_{\vec{p}}s . \quad (3.24)$$

This restriction may be written as

$$\frac{(1-v_{\vec{p}}\mu)k}{1+v_{\vec{p}}} \le s \le \frac{(1-v_{\vec{p}}\mu)k}{1-v_{\vec{p}}} , \quad (3.25)$$

so that, denoting these extreme values by s_a and s_b, respectively, we have

$$I_A = (2\pi)^{-1}\int d\vec{p}\, n_{\vec{p}}(1+n_{\vec{p}})(1-\mu)^2 v_{\vec{p}}^{-1}\int_{s_a}^{s_b} ds\, s^{-1} . \quad (3.26)$$

Since $v_{\vec{p}} \approx 1$, the result is

$$I_A = (2\pi)^{-1}\int d\vec{p}\, n_{\vec{p}}(1+n_{\vec{p}})(1-\mu)^2\ln[(1+v_{\vec{p}})/(1-v_{\vec{p}})] . \quad (3.27)$$

Note that the quantities $v_{\vec{p}}$ and $n_{\vec{p}}$ are identical for $\vec{p}$ and all points in the Brillouin zone equivalent to $\vec{p}$ under the symmetry operations of the crystal. Thus the quantity $(1-\mu)^2$ may be replaced by its average under the symmetry operations. For cubic symmetry this average, denoted by brackets, is

$$\langle (1-\mu)^2\rangle = \tfrac{4}{3} . \quad (3.28)$$

For $v_{\vec{p}}$ we use Eq. (3.13), neglecting the angular dependence of $g(\hat{p})$:

$$v_{\vec{p}} = 1-3p^2 g(\hat{p}) \approx 1-3p^2 g_{av} . \quad (3.29)$$

Then we find

$$I_A = \tau^3(a|\ln\tau|+a') , \quad (3.30)$$

where

$$a=\tfrac{16}{9}\pi^2\ , \tag{3.31a}$$

$$a'=\tfrac{8}{9}\pi^2\ln\tfrac{40}{3}-\tfrac{16}{3}\frac{d}{dx}\left[\zeta(x)\Gamma(x+1)\right]_{x=2}\approx 16.1\ , \tag{3.31b}$$

where $\Gamma(x)$ and $\zeta(x)$ are the Γ and Riemann ζ functions, respectively.[71] Thus in regime A, $\Gamma_{\vec{k}}$ is given in the first Born approximation as

$$\Gamma_{\vec{k}}=2\omega_E S^{-2}\epsilon_{\vec{k}}^2\tau^3(2\pi)^{-3}(a|\ln\tau|+a')\ . \tag{3.32}$$

We conclude this subsection by making some comments to justify our method of calculation. First, we remark that the contribution to $\Gamma_{\vec{k}}$ from smaller values of p at which the energy surface coalesces into a single distorted ellipsoid can be shown to be negligible. Furthermore, the assumption that the dominant contributions come when p is a thermal momentum is justified *a posteriori* by Eq. (3.27). Thus the retention of only the leading order in k/p and q/p in Eq. (3.18) was justified. These arguments indicate that Eq. (3.32) is the correct asymptotic evaluation of Eq. (2.38) when the approximation of Eq. (3.16) is made. The one remaining point to consider is whether there might be terms in $\Gamma_{\vec{k}}$ of much higher order in τ but of lower order in k. Such terms would clearly not be of the form of Eq. (3.17) and would result from terms, if any, of higher order in p but lower order in k in $M_{22}(\vec{k},\vec{p},\vec{r},\vec{s})$. To discuss this question, it is necessary to study $M_{22}(\vec{k},\vec{p},\vec{r},\vec{s})$ when k is small and the other momenta are only restricted by energy conservation but otherwise are arbitrarily large (this case is also relevant for the classical regime discussed in Sec. VI). Using the asymptotic forms in Appendix A, Eqs. (A2), (A5), (A8), and (A11), we have for small k, but for p and q finite,

$$\Phi^{(1)}_{\vec{k},\vec{p},\vec{r},\vec{s}}\sim(\epsilon_{\vec{p}}-\epsilon_{\vec{r}}-\epsilon_{\vec{s}})\ , \tag{3.33a}$$

$$l_{\vec{k}}^2\Phi^{(1)}_{\vec{r},\vec{s},\vec{k},\vec{p}}\sim\text{const}\ , \tag{3.33b}$$

$$\Phi^{(4)}_{\vec{k},-\vec{r},\vec{s},-\vec{p}}\sim(\epsilon_{\vec{r}}+\epsilon_{\vec{s}}-\epsilon_{\vec{p}})\ , \tag{3.33c}$$

$$l_{\vec{k}}^2\Phi^{(4)}_{\vec{s},-\vec{p},\vec{k},-\vec{r}}\sim\text{const}\ , \tag{3.33d}$$

$$\Phi^{(4)}_{\vec{k},-\vec{s},\vec{r},-\vec{p}}\sim(\epsilon_{\vec{s}}+\epsilon_{\vec{r}}-\epsilon_{\vec{p}})\ , \tag{3.33e}$$

$$l_{\vec{k}}^2\Phi^{(4)}_{\vec{r},-\vec{p},\vec{k},-\vec{s}}\sim\text{const}\ . \tag{3.33f}$$

Inserting these asymptotic forms into Eq. (2.39), we see that $M_{22}(\vec{k},\vec{p},\vec{r},\vec{s})$ for small k is of order $(\epsilon_{\vec{s}}+\epsilon_{\vec{r}}-\epsilon_{\vec{p}})$. On resonance,[72] we thus conclude that $M_{22}(\vec{k},\vec{p},\vec{r},\vec{s})$ is of order k for small k. Since $\Gamma_{\vec{k}}\sim\epsilon_{\vec{k}}M_{22}$, we see that there can be no terms in $\Gamma_{\vec{k}}$ of lower order in k than k^2 regardless of their order in τ.

C. Regime B

To evaluate $\Gamma_{\vec{k}}$ within the lowest Born approximation in regime B, we again start from Eq. (2.38):

$$\Gamma_{\vec{k}}=(\pi\omega_E/8S^2)(2\pi)^{-6}(1-e^{-\beta E_{\vec{k}}})\int d\vec{p}\int d\vec{q}\,\times n_{\vec{p}}(1+n_{\vec{r}})(1+n_{\vec{s}})M_{22}(\vec{k},\vec{p},\vec{r},\vec{s})\times\delta(2\epsilon_{\vec{k}}+2\epsilon_{\vec{p}}-2\epsilon_{\vec{r}}-2\epsilon_{\vec{s}})\ . \tag{3.34}$$

In this regime we may treat the scattering surface as a single ellipsoid. Also, since $k\ll\tau$, we write

$$1-e^{-\beta E_{\vec{k}}}\approx\beta E_{\vec{k}}=2\epsilon_{\vec{k}}/\tau\ . \tag{3.35}$$

On the energy shell, i.e., when $\epsilon_{\vec{k}}+\epsilon_{\vec{p}}=\epsilon_{\vec{r}}+\epsilon_{\vec{s}}$, we have

$$(1+n_{\vec{s}})(1+n_{\vec{r}})=(1+n_{\vec{r}}+n_{\vec{s}})(1-e^{-\beta E_{\vec{r}}-\beta E_{\vec{s}}})^{-1} \tag{3.36a}$$

$$=(1+n_{\vec{r}}+n_{\vec{s}})(1-e^{-\beta E_{\vec{k}}-\beta E_{\vec{p}}})^{-1} \tag{3.36b}$$

$$\approx(1+n_{\vec{r}}+n_{\vec{s}})(1+n_{\vec{p}})\ . \tag{3.36c}$$

Since the rest of the integrand is symmetric in $\vec{r}$ and $\vec{s}$, we may replace the factor $1+n_{\vec{r}}+n_{\vec{s}}$ by $1+2n_{\vec{s}}$. Inserting the long-wavelength approximation for M_{22} given in Eq. (3.18), we obtain the result

$$\Gamma_{\vec{k}}=\tfrac{1}{2}(\omega_E/S^2\tau)(2\pi)^{-3}\epsilon_{\vec{k}}^2 I_B\ , \tag{3.37}$$

where

$$I_B=(2\pi)^{-2}\int d\vec{p}\,n_{\vec{p}}(1+n_{\vec{p}})\int d\vec{s}\,(1+2n_{\vec{s}})(sp\,|\vec{p}+\vec{k}-\vec{s}|)^{-1}\times\delta(2\epsilon_{\vec{k}}+2\epsilon_{\vec{p}}-2\epsilon_{\vec{r}}-2\epsilon_{\vec{s}})\left[p^2(\hat{k}\cdot\hat{p}-1)^2+s^2(\hat{k}\cdot\hat{s}-1)^2+r^2(\hat{k}\cdot\hat{r}-1)^2\right]\ . \tag{3.38}$$

To evaluate the integral I_B we write

$$I_B=I_B^{(1)}+I_B^{(2)}\ , \tag{3.39}$$

where $I_B^{(1)}$ is the contribution in Eq. (3.38) from $s<s_0$ and $I_B^{(2)}$ is that from $s>s_0$, where $s_0=(kp)^{1/2}$.

In $I_B^{(1)}$ we have $s\le s_0\ll p$ and $k\ll p$, so that we neglect k and s in comparison to p. Then, using the notation of Eq. (3.22), we obtain an expression analogous to Eq. (3.23):

$$I_B^{(1)}=\pi^{-1}\int d\vec{p}\,n_{\vec{p}}(1+n_{\vec{p}})(\hat{k}\cdot\hat{p}-1)^2\int_0^{s_0}s\,ds\,(1+2n_{\vec{s}})\times\int_{-1}^{1}d\nu\,\delta(k-s+v_{\vec{p}}(\nu s-\mu k))\ , \tag{3.40a}$$

so that

$$I_B^{(1)}=2(\tau/\pi)\int d\vec{p}\,n_{\vec{p}}(1+n_{\vec{p}})(\hat{k}\cdot\hat{p}-1)^2\int_{s_a}^{s_0}ds\,(v_{\vec{p}}s)^{-1}\ , \tag{3.40b}$$

which is comparable to Eq. (3.26). Here we assume that $s_b=k(1-v_{\vec{p}}\mu)/(1-v_{\vec{p}})$ is larger than s_0, which is true except for an infinitesimal range of phase space, since $v_{\vec{p}}\approx 1$. Thus

$$I_B^{(1)}=\tfrac{32}{9}\tau\int_0^\infty p^2\,dp\,n_p(1+n_p)\left[1+3\ln(s_0/k)\right]\ , \tag{3.41}$$

where

$$n_p=(e^{p/\tau}-1)^{-1}\ . \tag{3.42}$$

In $I_B^{(2)}$ we have $k\ll s_0\le s$, so that k may be neglected in comparison to either s or p. Also we can perform the average over the orientation of $\vec{k}$ as in Eq. (3.28). In this way we obtain

$$I_B^{(2)} = (3\pi^2)^{-1} \int d\vec{p}\, n_{\vec{p}}\,(1+n_{\vec{p}}) \int d\vec{s}(1+2n_{\vec{s}})\,(ps\,|\vec{p}-\vec{s}\,|)^{-1} \times \delta(p+\xi-s-|\vec{p}-\vec{s}\,|)\,(p^2+s^2+|\vec{p}-\vec{s}\,|^2)\ . \tag{3.43}$$

Here we include the positive infinitesimal $\xi \equiv k - \vec{k}\cdot\vec{v}_{\vec{p}-\vec{s}} > 0$ in order to resolve the indeterminacy in the vanishing of the argument of the δ function. The integration is readily performed using

$$d\vec{p}\, d\vec{s} = 8\pi^2 p\, dp\, s\, ds\, r\, dr\ , \tag{3.44}$$

and the result to lowest order in k/τ is

$$I_B^{(2)} = \tfrac{8}{3}\int_k^\infty dp \int_{s_0}^p ds \int_{p-s}^{p+s} dr\, n_p(1+n_p)\,(1+2n_s) \times \delta(p+\xi-s-r)\,(2p^2+2s^2-2ps)\ . \tag{3.45}$$

We write the evaluation of this expression in the form

$$I_B^{(2)} = \tfrac{8}{3}\int_k^\infty dp \int_k^p ds\,(1+2n_s)\,n_p\,(1+n_p)\,(2p^2+2s^2-2ps) - \tfrac{16}{3}\int_k^\infty dp\, n_p\,(1+n_p)\int_k^{s_0}(2\tau/s)\,p^2\,ds\ , \tag{3.46}$$

from which we find

$$I_B = I_B^{(1)} + I_B^{(2)} = \tfrac{16}{3}\int_k^\infty dp \int_k^p ds\,(1+2n_s)\,n_p(1+n_p) \times (p^2+s^2-ps) + \tfrac{32}{9}\tau\int_k^\infty p^2 n_p(1+n_p)\,dp\ . \tag{3.47}$$

Explicit evaluation of these integrals to lowest order in τ leads to

$$I_B = \tfrac{16}{3}\,\tau^4[b\ln(\tau/k)+b'] , \tag{3.48}$$

where[71]

$$b = 4\zeta(2) = \tfrac{2}{3}\pi^2 = 6.58\ , \tag{3.49a}$$

$$b' = \tfrac{16}{3}\zeta(2) - 5\zeta(3) = 2.76\ , \tag{3.49b}$$

so that the first Born result for $\Gamma_{\vec{k}}$ in regime B is

$$\Gamma_{\vec{k}} = (8\omega_E/3S^2)\,(2\pi)^{-3}\,\epsilon_{\vec{k}}^2\,\tau^3\,|\,b\ln(\tau/k)+b'\,|\ . \tag{3.50}$$

The discussion to justify the above calculation is quite similar to that given in Sec. III B, and hence will be omitted. We note that the leading terms in the asymptotic forms for $\Gamma_{\vec{k}}$, (3.32) and (3.50), become equal to each other if one sets $k \approx \tau^3$ in the two expressions.

D. Regime C

To evaluate $\Gamma_{\vec{k}}$ within the first Born approximation in regime C, we again start from Eq. (2.38). Here, since $k \gg \tau$, we have

$$1 - e^{-\beta E_{\vec{k}}} \approx 1\ . \tag{3.51}$$

Furthermore, since $k \gg \tau$, we may also drop the exponential term in Eq. (3.36b), so that on the energy shell

$$(1+n_{\vec{r}})\,(1+n_{\vec{s}}) \approx 1 + n_{\vec{r}} + n_{\vec{s}}\ . \tag{3.52}$$

We again use the symmetry between $\vec{r}$ and $\vec{s}$ to replace the factor $1+n_{\vec{r}}+n_{\vec{s}}$ by $1+2n_{\vec{s}}$. Use of the long-wavelength expression for M_{22} given in Eq. (3.18) yields

$$\Gamma_{\vec{k}} = \tfrac{1}{4}\omega_E S^{-2}(2\pi)^{-3}\,\epsilon_{\vec{k}}\, I_C\ , \tag{3.53}$$

where

$$I_C = (2\pi)^{-2}\int d\vec{p}\, n_{\vec{p}} \int d\vec{s}\,(1+2n_{\vec{s}})\,(spr)^{-1} \times \delta(2\epsilon_{\vec{k}}+2\epsilon_{\vec{p}}-2\epsilon_{\vec{s}}-2\epsilon_{\vec{r}})\,[\,p^2(\hat{k}\cdot\hat{p}-1)^2 + p^2r^2k^{-2}(\hat{r}\cdot\hat{p}-1)^2 + p^2s^2k^{-2}(\hat{p}\cdot\hat{s}-1)^2]\ . \tag{3.54}$$

As before, we write

$$I_C = I_C^{(1)} + I_C^{(2)}\ , \tag{3.55}$$

where $I_C^{(1)}$ is the contribution in Eq. (3.54) from $s \le s_0$ and $I_C^{(2)}$ is that from $s \ge s_0$, where $s_0 = (kp)^{1/2}$. In $I_C^{(1)}$ we have $s \le s_0 \ll k$ and $p \ll k$, so that we neglect p and s in comparison to k. Using the notation $\nu' \equiv \hat{s}\cdot\hat{k}$, we then have to lowest order in p/k and s/k,

$$I_C^{(1)} = (2\pi k)^{-1}\int d\vec{p}\int_0^{s_0} s\,ds\, n_{\vec{p}}\,(1+2n_s)\,2p\,(1-\mu)^2 \times \int_{-1}^1 \delta(p-s+v_{\vec{k}}(s\nu'-p\mu))\,d\nu'\ . \tag{3.56}$$

In analogy with Eq. (3.41), we obtain

$$I_C^{(1)} = (2\pi k)^{-1}\int d\vec{p}\,2pn_{\vec{p}}\,(1-\mu)^2\int_{(1/2)p(1-\mu v_{\vec{k}})}^{s_0}(1+2n_s)\,ds\ . \tag{3.57}$$

Thus we conclude that $I_C^{(1)}$ is of order

$$I_C^{(1)} \sim \tau^5 k^{-1}\ln k\ , \tag{3.58}$$

which is negligible in comparison to what we shall find for $I_C^{(2)}$.

In $I_C^{(2)}$ we have $s \ge s_0 \gg p$, so that we neglect p in comparison to k and s. This procedure yields

$$I_C^{(2)} = (2\pi k^2)^{-1}\int d\vec{p}\int_{s_0}^k s\,ds\int_{-1}^1 d\nu'\, n_{\vec{p}}\,p^{-1}\,|\vec{k}-\vec{s}\,|^{-1} \times \delta(k+\xi-s-|\vec{k}-\vec{s}\,|)\,[\,p^2k^2(\hat{p}\cdot\hat{k}-1)^2 + (\vec{p}\cdot\vec{k}-\vec{p}\cdot\vec{s}-p\,|\vec{k}-\vec{s}\,|)^2 + p^2s^2(\hat{p}\cdot\hat{s}-1)^2]\ , \tag{3.59}$$

where ξ is an infinitesimal as in Eq. (3.43). The average over the orientations of $\vec{p}$ can be performed as in Eq. (3.28), and we use

$$d\vec{p}\,d\nu' = 4\pi p^2\,dp\,t\,dt\,(ks)^{-1}\ , \tag{3.60}$$

where $\vec{t} = \vec{k} - \vec{s}$, from which

$$I_C^{(2)} = \frac{8}{3k^3}\int_0^\infty p^3 n_p\,dp\int_{s_0}^k ds\,(2k^2+2s^2-2ks) \times \int_{k-s}^{k+s}\delta(k+\xi-s-t)\,dt\ . \tag{3.61}$$

The s and t integrals yield

$$I_C^{(2)} = \tfrac{40}{9} \int_0^\infty n_p p^2 \, dp \ , \qquad (3.62)$$

so that

$$I_C = \tfrac{80}{3} \tau^4 \zeta(4) = \tfrac{8}{27} \pi^4 \tau^4 \ . \qquad (3.63)$$

Thus the first Born approximation for the decay rate in regime C is

$$\Gamma_{\vec{k}} = \omega_E (\pi/108 S^2) \epsilon_{\vec{k}} \tau^4 \ . \qquad (3.64)$$

E. Regime D

The calculations for regime D are quite similar to those for regime A except that the roles of $\vec{k}$ and $\vec{p}$ are reversed. Here, for the momenta p of interest, i.e., for $p \approx \tau$, the scattering surface consists of two disjoint pieces. In this case, however, we need keep only the terms with a single occupation number, because the other terms will lead to a more rapid temperature dependence. Then writing $\Gamma_{\vec{k}}$ as twice the contribution from the scattering surface near $q=0$, we obtain

$$\Gamma_{\vec{k}} = (\pi\omega_E/4S^2)(2\pi)^{-6} \int d\vec{p} \int d\vec{q}\, n_{\vec{p}} M_{22}(\vec{k}, \vec{p}, \vec{r}, \vec{s}) \times \delta(p - r + (\vec{r} - \vec{p}) \cdot \vec{v}_{\vec{k}}) \ . \qquad (3.65)$$

Here, since k is the dominant momentum, we use Eq. (A23) to transform Eq. (3.16) for M_{22} at long wavelength into the form

$$M_{22}(\vec{k}, \vec{p}, \vec{r}, \vec{s}) = 2(\epsilon_{\vec{k}}\epsilon_{\vec{p}}\epsilon_{\vec{r}}\epsilon_{\vec{s}})^{-1} [\epsilon_{\vec{k}}^2 \epsilon_{\vec{p}}^2 (\hat{k}\cdot\hat{p} - 1)^2 + \epsilon_{\vec{r}}^2 \epsilon_{\vec{p}}^2 (\hat{r}\cdot\hat{p} - 1)^2 + \epsilon_{\vec{s}}^2 \epsilon_{\vec{p}}^2 (\hat{s}\cdot\hat{p} - 1)^2] \qquad (3.66a)$$

$$= 4\epsilon_{\vec{p}}\epsilon_{\vec{r}}^{-1} (\hat{k}\cdot\hat{p} - 1)^2 \ , \qquad (3.66b)$$

where we have used the conditions $k \gg p$ and $k \gg q$ in the last step. Use of this form in Eq. (3.65) leads to the result

$$\Gamma_{\vec{k}} = (\omega_E/4S^2)(2\pi)^{-3} I_D \ , \qquad (3.67)$$

where

$$I_D = (2\pi^2)^{-1} \int d\vec{p} \int d\vec{r}\, n_{\vec{p}} (\hat{k}\cdot\hat{p} - 1)^2 p r^{-1} \times \delta(p - r + \vec{r}\cdot\vec{v}_{\vec{k}} - \vec{p}\cdot\vec{v}_{\vec{k}}) \ . \qquad (3.68)$$

To evaluate I_D, we note that the r integration here is similar to the s integration in Eq. (3.21). Thus

$$I_D = \pi^{-1} \int d\vec{p}\, n_{\vec{p}} (1-\mu)^2 p \int_{r_a}^{r_b} v_{\vec{k}}^{-1}\, dr \ , \qquad (3.69)$$

where again $\mu \equiv \hat{k}\cdot\hat{p}$ and

$$r_a = p(1 - \mu v_{\vec{k}})/(1 + v_{\vec{k}}) \ , \qquad (3.70a)$$

$$r_b = p(1 - \mu v_{\vec{k}})/(1 - v_{\vec{k}}) \ . \qquad (3.70b)$$

Thus the r integration yields

$$I_D = \pi^{-1} \int d\vec{p}\, n_{\vec{p}} p^2 (1-\mu)^2 \left(\frac{1 - v_{\vec{k}}\mu}{1 - v_{\vec{k}}} - \frac{1 - v_{\vec{k}}\mu}{1 + v_{\vec{k}}} \right) v_{\vec{k}}^{-1} \ . \qquad (3.71)$$

Noting that $v_{\vec{k}} \approx 1$, and also using the average under cubic symmetry

$$\langle (1-\mu)^3 \rangle = 2 \ , \qquad (3.72)$$

we find

$$I_D = (2/\pi) \int d\vec{p}\, n_{\vec{p}} p^2 (1 - v_{\vec{k}})^{-1} = 64 \zeta(5) \tau^5 / k^2 g(\hat{k}) \ . \qquad (3.73)$$

Thus the first Born result for the decay rate in regime D is

$$\Gamma_{\vec{k}} = (\omega_E / 2S^2\pi^3) \tau^5 \zeta(5) [g(\hat{k}) \epsilon_{\vec{k}}^2]^{-1} \ , \qquad (3.74)$$

with $g(\hat{k})$ given in Eq. (3.8) and $\zeta(5) = 1.037$.[71]

IV. SELF-CONSISTENCY OF BORN APPROXIMATION

In the foregoing calculation of the damping of long-wavelength magnons we have treated the intermediate thermal magnons as free particles, i.e., we have completely neglected their damping. On the other hand, by interpolating between the results of regimes B and C we may estimate the damping of a thermal ($H_E \epsilon_{\vec{k}} \approx \hbar\omega \approx k_B T$) magnon to be of order $\Gamma_{\text{th}} \approx \omega_E S^{-2} \tau^5$. It is quite possible that when the frequency ω of the incoming particle becomes less than this decay rate of thermal magnons, the Born approximation will cease to give a correct estimate of the damping $\Gamma_{\vec{k}}$. In particular, a new "hydrodynamic" regime might set in, when $\omega \ll \Gamma_{\text{th}}$, as it does in phonon systems such as liquid helium at low temperatures,[31–33] or a crystal.[31, 36–38] It is thus important to test the stability of the Born approximation with respect to the inclusion of damping in intermediate states.

Such a test seems particularly necessary, in view of the fact that the calculations of Sec. II depend critically on the precise form of the surface generated by the energy-conserving δ function. Indeed, using Eq. (A23), we may write the matrix element in Eq. (3.18) as

$$\tfrac{1}{2} M_{22}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = (\epsilon_1\epsilon_2\epsilon_3\epsilon_4)^{-1} \{ \epsilon_1^2\epsilon_2^2 (\hat{k}_1\cdot\hat{k}_2 - 1)^2 + \epsilon_1^2\epsilon_3^2 (\hat{k}_1\cdot\hat{k}_3 - 1)^2 + \epsilon_1^2\epsilon_4^2 (\hat{k}_1\cdot\hat{k}_4 - 1)^2 - \Delta\epsilon_{12} [\epsilon_1\epsilon_2 (1 - \hat{k}_1\cdot\hat{k}_2)(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4) - \epsilon_1\epsilon_3 (1 - \hat{k}_1\cdot\hat{k}_3)\Delta\epsilon_{14} - \epsilon_1\epsilon_4 (1 - \hat{k}_1\cdot\hat{k}_4)\Delta\epsilon_{13}] \} \ . \qquad (4.1)$$

In the limit when $\epsilon_{\vec{k}} \equiv \epsilon_1 \to 0$, M_{22} has one term proportional to ϵ_1, which was retained previously, plus another term proportional to $\Delta\epsilon_{12} \to \epsilon_2 - \epsilon_3 - \epsilon_4 = \epsilon_{\vec{p}} - \epsilon_{\vec{s}} - \epsilon_{\vec{p}-\vec{s}}$ which remains finite when $\epsilon_1 \to 0$.

This term did not contribute in the calculation of Sec. III B because of the δ-function condition, which assumed the intermediate magnons to be on the energy shell, and placed the incident magnon on resonance[72] ($\Delta\epsilon_{12}=0$). When the damping of the intermediate magnons is included, however, the δ-function condition must be relaxed, and the finite matrix element M_{22} for $\epsilon_{\vec{t}}\to 0$ could lead to a correction in the decay rate $\Gamma_{\vec{k}}$ of relative order $(\Gamma_{th}/\omega_E\epsilon_{\vec{k}})\gg 1$.

In fact, the δ-function condition must already be modified in Born approximation, even without including damping in intermediate states, if the imaginary part of the self-energy is calculated off-resonance, i.e., for an *incoming* magnon whose frequency and wave number are not related by the resonance condition $\hbar\omega=H_E\epsilon_{\vec{k}}$. In that case, the argument of the energy-conserving δ function is not $\Delta\epsilon_{12}=\epsilon_{\vec{k}}+\epsilon_{\vec{p}}+\epsilon_{\vec{s}}-\epsilon_{\vec{r}}$, but rather $\tilde{\omega}+\epsilon_{\vec{p}}-\epsilon_{\vec{r}}-\epsilon_{\vec{s}}$, where

$$\tilde{\omega}=\hbar\omega/H_E=\omega/\omega_E\ . \tag{4.2}$$

The quantity $\Delta\epsilon_{12}=\epsilon_{\vec{k}}-\tilde{\omega}$ will not vanish, and hence must be retained in M_{22} in calculating the decay rate, which we now proceed to do.

A. Born Approximation for Off-Resonance Magnons

When the argument of the δ function is modified, the scattering surfaces are no longer exactly as described in Sec. III. In particular, certain processes which were excluded from the previous calculation because they could not satisfy "energy conservation" must now be taken into account. Specifically, we must include diagrams such as in Figs. 2(a) and 2(b), containing vertices in which three magnons come in and one goes out, and vice versa.[73] In the Born approximation, the contribution from the diagrams of Fig. 1 considered earlier, and of Figs. 2(a) and 2(b), may be written, after some manipulation, in the form

$$\begin{aligned}\Gamma(\vec{k},\omega)\equiv\hbar^{-1}\Sigma''(\vec{k},\omega)&=\tfrac{1}{4}\pi\,\omega_E S^{-2}(1-e^{-\beta\hbar\omega})(2\pi)^{-6}\\ &\times\int d\vec{p}\int_{s\approx 0}d\vec{s}\,n_{\vec{p}}(1+n_{\vec{s}})(1+n_{\vec{r}})\\ &\times[\delta(2\tilde{\omega}+2\epsilon_{\vec{p}}-2\epsilon_{\vec{s}}-2\epsilon_{\vec{r}})M_{22}(\vec{k},\vec{p},\vec{s},\vec{r})\\ &+e^{\beta\hbar\omega}\delta(-2\tilde{\omega}+2\epsilon_{\vec{p}}-2\epsilon_{\vec{r}}-2\epsilon_{\vec{s}})M_{31}(\vec{k},\vec{p},\vec{s},\vec{r})]\ ,\end{aligned}\tag{4.3}$$

where M_{22} at long wavelengths is given by Eq. (3.16), and for the diagrams of Figs. 2(a) and 2(b), we have the corresponding matrix element M_{31} given as[70]

$$\begin{aligned}M_{31}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)=8l_1^2l_2^2l_3^2l_4^2[&\Phi^{(6)}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\Phi^{(5)}(\vec{k}_3,\vec{k}_4,\vec{k}_1,\vec{k}_2)+\Phi^{(3)}(-\vec{k}_1,\vec{k}_4,\vec{k}_2,-\vec{k}_3)\Phi^{(2)}(\vec{k}_2,-\vec{k}_3,-\vec{k}_1,\vec{k}_4)\\ &+\Phi^{(3)}(-\vec{k}_1,\vec{k}_3,\vec{k}_2,-\vec{k}_4)\Phi^{(2)}(\vec{k}_2,-\vec{k}_4,-\vec{k}_1,\vec{k}_3)]\ ,\end{aligned}\tag{4.4}$$

which at long wavelengths becomes

$$\tfrac{1}{2}M_{31}\approx-[(1+\hat{k}_1\cdot\hat{k}_2)(1-\hat{k}_3\cdot\hat{k}_4)+(1+\hat{k}_1\cdot\hat{k}_3)(1-\hat{k}_2\cdot\hat{k}_4)+(1+\hat{k}_1\cdot\hat{k}_4)(1-\hat{k}_2\cdot\hat{k}_3)]\ . \tag{4.5}$$

If we restrict ourselves to regime A, in which $k\ll\tau^3$, we may follow the procedure of Sec. III B and expand the integrand in powers of $\epsilon_{\vec{s}}/\epsilon_{\vec{p}}$ and $\epsilon_{\vec{k}}/\epsilon_{\vec{p}}$. Then the matrix elements in Eq. (4.3) may be approximated as

$$\tfrac{1}{4}M_{22}(\vec{k},\vec{p},\vec{s},\vec{r})\approx(\epsilon_{\vec{k}}/\epsilon_{\vec{s}})\{(1-\mu)^2-[(\Delta+\epsilon_{\vec{k}})/\epsilon_{\vec{k}}](1-\mu)\}\ , \tag{4.6a}$$

$$\tfrac{1}{4}M_{31}(\vec{k},\vec{p},\vec{s},\vec{r})\approx(\epsilon_{\vec{k}}/\epsilon_{\vec{s}})\{(1+\mu)^2-[(-\Delta+\epsilon_{\vec{k}})/\epsilon_{\vec{k}}](1+\mu)\}\ , \tag{4.6b}$$

where

$$2\Delta=-s+v_{\vec{p}}s\nu-v_{\vec{p}}k\mu\ , \tag{4.7}$$

and we have used Eq. (A23) and the notation of Sec. III B for the coordinates. In analogy with Eq. (3.20), let us write Eq. (4.3) as

$$\Gamma(\vec{k},\omega)=2\omega_E S^{-2}(2\pi)^{-3}\rho\epsilon_{\vec{k}}^2K(\epsilon_{\vec{k}},\rho)\ , \tag{4.8}$$

where

$$\begin{aligned}K(\epsilon_{\vec{k}},\rho)=(2\pi)^{-2}\int d\vec{p}\int d\vec{s}\,n_{\vec{p}}(1+n_{\vec{p}})s^{-2}\{&\delta(2\epsilon_{\vec{k}}\rho+2\Delta)[(1-\mu)^2-(1+\Delta/\epsilon_{\vec{k}})(1-\mu)]\\ &+\delta(2\epsilon_{\vec{k}}\rho-2\Delta)[(1+\mu)^2-(1-\Delta/\epsilon_{\vec{k}})(1+\mu)]\}\ ,\end{aligned}\tag{4.9}$$

with

$$\rho = \hbar\omega / E_{\vec{k}} = \tilde{\omega}/\epsilon_{\vec{k}} = \omega/\omega_E \epsilon_{\vec{k}} \ . \tag{4.10}$$

The integral in Eq. (4.9) is performed in Appendix D in the limit $\epsilon_{\vec{k}} \to 0$, with ρ finite, and the result is

$$K(0, \rho) = \tfrac{1}{4}(1+3\rho) I_A \ , \tag{4.11}$$

where I_A is given in Eq. (3.30). On resonance, we have $\rho = 1$ and the previous result is recovered.

B. Inclusion of Damping in Intermediate States

In order to take into account the effect of the finite lifetime of thermal magnons, we shall rewrite the mass operator with fully dressed propagators in the intermediate states. According to the rules of Refs. 53 and 54, we have

$$\begin{aligned}\Gamma(\mathrm{k}, \omega) = {} & \tfrac{1}{16}\pi\omega_E S^{-2}(1 - e^{-\beta\hbar\omega}) \int_{-\infty}^{+\infty} \frac{d\tilde{\omega}_2}{2\pi} \int_{-\infty}^{+\infty} \frac{d\tilde{\omega}_3}{2\pi} \\ & \times \int_{-\infty}^{+\infty} \frac{d\tilde{\omega}_4}{2\pi} n(\tilde{\omega}_2)[1+n(\tilde{\omega}_3)][1+n(\tilde{\omega}_4)](2\pi)^{-6} \\ & \times \int d\vec{p} \int d\vec{q}\, \tilde{A}(\vec{p}, \tilde{\omega}_2)\tilde{A}(\vec{s}, \tilde{\omega}_3)\tilde{A}(\vec{r}, \tilde{\omega}_4) \\ & \times [\delta(\tilde{\omega} + \tilde{\omega}_2 - \tilde{\omega}_3 - \tilde{\omega}_4) M_{22}(\vec{k}, \vec{p}, \vec{s}, \vec{r}) \\ & + e^{\beta\hbar\omega}\delta(-\tilde{\omega} + \tilde{\omega}_2 - \tilde{\omega}_3 - \tilde{\omega}_4) M_{31}(\vec{k}, \vec{p}, \vec{s}, \vec{r})] \ ,\end{aligned} \tag{4.12}$$

where $\tilde{\omega}_i = \omega_i/\omega_E$ (tildes usually denote dimensionless quantities), and

$$\tilde{A}(\vec{k}_i, \tilde{\omega}_i) = 2H_E \lim_{\eta \to 0^+} \mathrm{Im}\, G_{\alpha\alpha}(\vec{k}_i, \omega_i - i\eta) \tag{4.13}$$

is the spectral-weight function. In deriving Eq. (4.12) we have used the relation

$$\tilde{A}_{\alpha\alpha}(\vec{k}_i, \tilde{\omega}_i) = -\tilde{A}_{\beta\beta}(\vec{k}_i, -\tilde{\omega}_i) \ , \tag{4.14}$$

which follows from the general symmetry properties, Eq. (2.24), and we have neglected the off-diagonal term $\tilde{A}_{\alpha\beta}$, which would lead to corrections of higher order in the density of spin deviations, i.e., in the temperature or in $1/zS$. For undamped spin waves we have

$$\tilde{A}(\vec{k}_i, \tilde{\omega}_i) = 2\pi\delta(\tilde{\omega} - \epsilon_i) \ , \tag{4.15}$$

in which case Eq. (4.12) reduces to Eq. (4.3).

In order to evaluate the decay rate we must solve Eq. (4.12) self-consistently, since the spectral weights are themselves functions of the mass operator Σ. Let us take $\tilde{A}(\vec{k}_i, \tilde{\omega}_i)$ to be a function sharply peaked about $\tilde{\omega}_i = \epsilon_i$, whose dominant contribution occurs within a characteristic width γ_i which is assumed to be of order $S^{-2}\tau^5$ for a momentum k_i of order τ. In particular, we are assuming here that any collective effects of the magnon interactions such as possible bound states[74–76] or "second magnons"[35, 77–79] may be neglected in the spectral-weight function at long wavelenghts.[80] If we now perform the $\tilde{\omega}_i$ integrals in Eq. (4.12), we may replace the frequencies $\tilde{\omega}_i$ in the occupation numbers by the corresponding energies ϵ_i, since the occupation numbers vary on a scale $\tau \gg \gamma_i$. The $\tilde{\omega}_i$ integrals then reduce to

$$\begin{aligned}&\int_{-\infty}^{+\infty} \frac{d\tilde{\omega}_2}{2\pi} \int_{-\infty}^{+\infty} \frac{d\tilde{\omega}_3}{2\pi} \int_{-\infty}^{+\infty} \frac{d\tilde{\omega}_4}{2\pi} \tilde{A}(\vec{p}, \tilde{\omega}_2)\tilde{A}(\vec{s}, \tilde{\omega}_3)\tilde{A}(\vec{r}, \tilde{\omega}_4) \\ &\times \delta(\tilde{\omega} + \tilde{\omega}_2 - \tilde{\omega}_3 - \tilde{\omega}_4) \equiv \varphi(\tilde{\omega} + \epsilon_{\vec{p}} - \epsilon_{\vec{r}} - \epsilon_{\vec{s}}) \ .\end{aligned} \tag{4.16}$$

The function $\varphi(x)$ is again sharply peaked at $x = 0$, with a width equal to the sum of the widths γ_i. This function may be thought of as a normalized probability[81, 82] which smears the scattering surface by an amount

$$\gamma \equiv \omega_E^{-1}\Gamma_{\mathrm{th}} \approx S^{-2}\tau^5 \ . \tag{4.17}$$

As always, our main interest centers on the domain $p \approx \tau$. In that case, it is clear from the discussion in Sec. III B that in Eq. (4.12) we have $q \ll p$ and, therefore, also $s \ll p$ and $r \approx p$. In the limit when k and q vanish, it is easily verified from Eq. (4.16) that $\varphi(x)$ is an even function of x independent of the shape of the spectral-weight function, assuming the width γ_i to vanish at zero momentum. Hence, we conclude that for the important values of p (i.e., $p \approx \tau$) $\varphi(x)$ is essentially an even function of x. This argument justifies writing

$$\int_{-\infty}^{+\infty} \varphi(x)\, dx = 1 \ , \tag{4.18a}$$

$$\int_{-\infty}^{+\infty} x^2 \varphi(x)\, dx = \chi(\vec{p})\gamma^2 \ , \tag{4.18b}$$

where $\chi(\vec{p})$ is a dimensionless function which is of order unity for $p \approx \tau$.

Restricting ourselves to small values of $\epsilon_{\vec{k}}$ and $\tilde{\omega}$ (in regime A we have $\epsilon_{\vec{k}} \ll \tau^3$), we may transform Eq. (4.12) to a form analogous to Eq. (4.8):

$$\Gamma(\vec{k}, \omega) = 2S^{-2}\omega_E \rho \epsilon_{\vec{k}}^2 (2\pi)^{-3}\tilde{K}(\epsilon_{\vec{k}}, \rho) \ , \tag{4.19}$$

where $\tilde{K}$ is given as

$$\begin{aligned}\tilde{K}(\epsilon_{\vec{k}}, \rho) = {} & (2\pi)^{-2}\int d\vec{p} \int d\vec{s}\, n_{\vec{p}}(1+n_{\vec{p}}) s^{-2} \int_{-\infty}^{+\infty} \varphi(\alpha)\, d\alpha \\ & \times \{\delta(2\epsilon_{\vec{k}}\rho + 2\Delta - \alpha)[(1-\mu)^2 - (1+\Delta/\epsilon_{\vec{k}})(1-\mu)] \\ & + \delta(-2\epsilon_{\vec{k}}\rho + 2\Delta - \alpha)[(1+\mu)^2 - (1-\Delta/\epsilon_{\vec{k}})(1+\mu)]\} \ .\end{aligned} \tag{4.20}$$

If we change μ to $-\mu$ in the second term in the curly bracket, and change variables in the α integrals, we find

$$\begin{aligned}\tilde{K}(\epsilon_{\vec{k}}, \rho) = {} & (2\pi)^{-2}\int d\vec{p} \int d\vec{s}\, n_{\vec{p}}(1+n_{\vec{p}}) s^{-2} \int_{-\infty}^{+\infty} d\alpha\, (1-\mu)\Delta^0 \\ & \times \delta(2\Delta^0 - \alpha)\{\epsilon_{\vec{k}}^{-1}[\varphi(\alpha - 2\epsilon_{\vec{k}}\rho + 2\epsilon_{\vec{k}}\mu) \\ & - \varphi(\alpha + 2\epsilon_{\vec{k}}\rho - 2\epsilon_{\vec{k}}\mu)]\} \ ,\end{aligned} \tag{4.21}$$

with

$$2\Delta^0 = 2\Delta + 2\epsilon_{\vec{k}}\,\mu = -s + v_{\vec{p}}\,s\nu \ . \tag{4.22}$$

In obtaining Eq. (4.21) we have dropped a term of relative order $\epsilon_{\vec{k}}(1-v_{\vec{p}})$, which is negligible for $p\approx\tau$. Similarly, we have set $v_{\vec{p}}\approx 1$ in the argument of the φ functions. In the limit $\epsilon_{\vec{k}}/\gamma \ll 1$ we may expand the curly bracket in Eq. (4.21) about $\epsilon_{\vec{k}}=0$, since the expansion is in powers of $(\epsilon_{\vec{k}}/\gamma)^2$, as is discussed in Appendix D. Keeping only the leading term, we find an expression independent of $\epsilon_{\vec{k}}$, and we can now set $\epsilon_{\vec{k}}$ equal to zero. The remaining integral is a simple one (see Appendix D) and the answer is once again

$$\tilde{K}(0,\ \rho) = \tfrac{1}{4}(1+3\rho)I_A \ , \tag{4.23}$$

to leading order in $(\epsilon_{\vec{k}}/\gamma)$. Thus we find identical results for $\gamma=0$ and for $\epsilon_{\vec{k}}/\gamma\ll 1$. In fact, Eq. (4.21) may be evaluated for finite values of $\epsilon_{\vec{k}}/\gamma$, if it is assumed that the probability function $\varphi(x)$ is an even function of x.[83] This evaluation is also given in Appendix D and again the result is the one in Eq. (4.23). Thus for arbitrary values of $\epsilon_{\vec{k}}/\gamma$, we have

$$\Gamma(\vec{k},\ \omega) = 2\omega_E S^{-2}(2\pi)^{-3} I_A\, \epsilon_{\vec{k}}^2\,\rho \times \tfrac{1}{4}(1+3\rho) \ . \tag{4.24}$$

Accordingly, we conclude that the Born approximation is self-consistent both on and off the energy shell, in that it is stable with respect to inclusion of damping in the intermediate states. In particular, although the matrix elements M_{22} and M_{31} remain finite for $\epsilon_{\vec{k}}\to 0$ [see Eq. (4.6)], the combination of M_{22} and M_{31} which occurs in the decay rate $\Gamma(\vec{k},\ \omega)$ [see Eqs. (4.12) and (4.20)] vanishes in the limit $\epsilon_{\vec{k}}\to 0$, $\tilde{\omega}\to 0$, with $\rho=\tilde{\omega}/\epsilon_{\vec{k}}$ finite. This means that, within the Born approximation, a long-wavelength magnon does interact weakly with other magnons, both on and off the energy shell. Near resonance, Eq. (4.20) yields a damping proportional to $\epsilon_{\vec{k}}^2$, a result in agreement with the predictions of hydrodynamics.[12,13] This behavior sets in when the energy $E_{\vec{k}} = H_E\epsilon_{\vec{k}}$ becomes much less than the curvature energy of thermal magnons $H_E\tau^3$, and remains unchanged when $E_{\vec{k}}$ becomes less than the average width $\hbar\Gamma_{\text{th}} \approx H_E S^{-2}\tau^5$ of thermal magnons. This point will be discussed further in Sec. VII C. Finally, we remark that the Holstein-Primakoff formalism might be expected not to lead to a self-consistent result in the regime $k\ll\gamma$, because in that formalism all matrix elements occur in positive combinations due to the Hermiticity of the Hamiltonian, and hence in lowest Born approximation no cancellation of terms of relative order $(\gamma/\epsilon_{\vec{k}})$ is possible. The Holstein-Primakoff formalism is discussed in detail in Appendix F.

V. HIGHER-ORDER PROCESSES

A. Introduction

In Sec. IV, we have carried out a calculation of the decay rate, taking into account second-order processes involving self-consistently damped magnons. In this section, we consider the contributions to the decay rate from higher-order processes. Although our analysis is not rigorous, it does suggest that these higher-order processes do not qualitatively affect the results of the lowest Born approximation. In order to avoid obscuring the general lines of our arguments, we have placed most of the computational details in Appendices G–I.

For finite wave vectors, the analysis is simple because contributions to the decay rate are negligible if they are of sufficiently high order in τ. Since, roughly speaking, each hole line introduces a factor τ^3, one can in this case restrict one's attention to diagrams with a minimum number of hole lines. Similarly, one can consider only diagrams with a fixed number of repeated scatterings, because processes which are of sufficiently high order in $1/z$ are also negligible for finite $\epsilon_{\vec{k}}$. Thus the lowest Born approximation is expected to give the leading term for finite wave vectors.

On the other hand, in the regime where $\epsilon_{\vec{k}}$ is the smallest energy in the problem, the above considerations do not apply, and the analysis is considerably more complex. We wish to show that even in this regime, higher-order processes lead to corrections to the decay rate which are small when $\epsilon_{\vec{k}}\to 0$ and the density of spin deviations remains small. This means that we must show that terms of relative order $\tau^5/\epsilon_{\vec{k}}$, for instance, do not occur in the perturbation series. Then the long-wavelength limit may be taken at fixed temperature, and the lowest Born approximation also represents the leading contribution in the limit $\epsilon_{\vec{k}}\ll\tau^5$. A rigorous proof of this assertion, even within perturbation theory, would involve examining the analytic properties of diagrams of arbitrary order. Although we do not have such a proof, we are able to argue that the various cancellations which are responsible for the long-wavelength behavior of the decay rate in the Born approximation, will also occur in higher orders. Thus the physical spin waves interact weakly at long wavelengths, and the decay rate agrees with the predictions of hydrodynamics.[13] The weakness of the interaction between antiferromagnetic magnons forms the basis of an analogy with the ferromagnet, considered in Sec. V C.

B. Analysis of Higher-Order Diagrams

In order to clarify the discussion of higher-order terms, let us reexamine the self-consistent calculation of Sec. IV, and try to identify those properties which are responsible for the result obtained there, namely,

$$\Gamma(\vec{k},\ \omega) = \omega_E S^{-2}\,\epsilon_{\vec{k}}^2 f(\rho)\,\tau^3\ln\tau \ . \tag{5.1}$$

Here $f(\rho)$ is a function of order unity, and throughout this section we consider the "long-wavelength limit" to be defined by $\epsilon_{\vec{k}} \ll \tau^{n_0}$, with $\rho = \tilde{\omega}/\epsilon_{\vec{k}}$ finite, and n_0 is an unknown fixed number. First, we note that one factor of $\epsilon_{\vec{k}}$ in Eq. (5.1) comes from a "detailed balance" term $(1 - e^{-\beta\hbar\omega}) \sim \beta\hbar\omega = \beta\hbar\omega_E \rho\epsilon_{\vec{k}}$ representing the difference between "forward" and "backward" processes (see Sec. IID and Appendix G). Thus we may write

$$\Gamma(\vec{k}, \omega) = (1 - e^{\beta\hbar\omega})\Gamma_{>}(\vec{k}, \omega) \sim \beta\hbar\omega_E \rho\epsilon_{\vec{k}}\Gamma_{>}(\vec{k}, \omega), \tag{5.2}$$

where $\Gamma_{>}(\vec{k}, \omega)$ is the decay rate for "forward" processes. The second essential feature in the calculation of Sec. IV is the cancellation, or interference, between the matrix elements M_{22} and M_{31} in Eq. (4.3), corresponding to the contributions from the diagrams in Figs. 1 and 2, respectively. This cancellation ensures that there are no constant terms in $\Gamma_{>}(\vec{k}, \omega)$ as $\epsilon_{\vec{k}} \to 0$, so that

$$\Gamma_{>}(\vec{k}, \omega) \sim \epsilon_{\vec{k}}^{-1}\Gamma(\vec{k}, \omega) \to 0 \quad \text{as } \epsilon_{\vec{k}} \to 0 . \tag{5.3a}$$

If we generalize the definition of $K(\epsilon_{\vec{k}}, \rho)$ in Eq. (4.8) and write

$$\Gamma(\vec{k}, \omega) = \omega_E \rho\epsilon_{\vec{k}}^2 K(\epsilon_{\vec{k}}, \rho), \tag{5.3b}$$

then the cancellation implies that

$$\rho\epsilon_{\vec{k}} K(\rho, \epsilon_{\vec{k}}) \to 0 \quad \text{as } \epsilon_{\vec{k}} \to 0 . \tag{5.3c}$$

In other words, the "dangerous" terms of order $\Delta/\epsilon_{\vec{k}}$ in Eq. (4.9) cancel to leading order in $\epsilon_{\vec{k}}$. Note that the property (5.3a) does not imply the validity of Eq. (5.1), since a decay rate of order $\epsilon_{\vec{k}}^{3/2}$, for example, would be consistent with Eq. (5.3) but not with Eq. (5.1). But if $\epsilon_{\vec{k}}^{-1}\Gamma(\vec{k}, \omega)$ is a *regular* function of $\epsilon_{\vec{k}}$, i.e., if $K(0, \rho)$ is finite, then we may write

$$\Gamma(\vec{k}, \omega) = \omega_E S^{-2}\epsilon_{\vec{k}}^2 g(\rho, \tau), \quad \epsilon_{\vec{k}} \to 0 . \tag{5.4}$$

This regularity in $\epsilon_{\vec{k}}$ is the third essential property of the calculation of Sec. IV and, as shown in Appendix D, it arises from the expansion of the spectral weight functions in powers of $\epsilon_{\vec{k}}/\gamma$, where γ is the effective width of thermal magnons. The fourth and final feature of the calculation in Sec. IV is the absence of large temperature renormalizations, when the damping of intermediate states is taken into account. This comes about because in the *absence* of damping in intermediate states, the matrix elements are proportional to the energy transfer [see Eq. (4.1)]. As we shall show below, this property is a result of the Hermiticity of the effective interactions on resonance.

To illustrate the significance of these four properties of the second-order calculation (viz., detailed balance, cancellation of matrix elements, regularity in $\epsilon_{\vec{k}}$, and absence of anomalous temperature renormalization), let us use them to estimate the decay rate which we have calculated explicitly in Secs. III and IV and in Appendix D. Let us first consider the decay rate for an incoming magnon which is on-resonance ($\tilde{\omega} = \epsilon_{\vec{k}}$), neglecting the damping of intermediate states. Then from Eq. (3.19) we see that $M_{22} \sim \epsilon_{\vec{k}}/\epsilon_{\vec{s}}$ which leads to the estimate

$$\Gamma_{\vec{k}} \sim (\beta\epsilon_{\vec{k}})(\epsilon_{\vec{k}}/\epsilon_{\vec{s}})(\tau^4\epsilon_{\vec{s}}) \tag{5.5a}$$

$$\sim \tau^3\epsilon_{\vec{k}}^2 . \tag{5.5b}$$

Here $(\beta\epsilon_{\vec{k}})$ is the detailed-balance factor, $(\epsilon_{\vec{k}}/\epsilon_{\vec{s}})$ comes from M_{22}, and $(\tau^4\epsilon_{\vec{s}})$ is the volume element in phase space when the occupation numbers are averaged over the scattering surface. For the decay rate off-resonance ($\tilde{\omega} \neq \epsilon_{\vec{k}}$), one finds terms with $\epsilon_{\vec{k}}$ replaced by $\tilde{\omega}$, which are of the same order when $\rho = \tilde{\omega}/\epsilon_{\vec{k}}$ is finite. For the decay rate when damping of intermediate states is allowed, we encounter terms in M_{22} and M_{31} of order $\gamma/\epsilon_{\vec{s}}$ [see Eq. (4.20)]. Then we estimate

$$\Gamma_{\vec{k}} \sim (\beta\epsilon_{\vec{k}})(\gamma/\epsilon_{\vec{s}})(\tau^4\epsilon_{\vec{s}})(\epsilon_{\vec{k}}/\gamma), \tag{5.6a}$$

$$\Gamma_{\vec{k}} \sim \tau^3\epsilon_{\vec{k}}^2 . \tag{5.6b}$$

The first term is the detailed-balance factor, the factor $\gamma/\epsilon_{\vec{s}}$ comes from the matrix element, $\tau^4\epsilon_{\vec{s}}$ comes from the volume element in phase space, and the final factor $\epsilon_{\vec{k}}/\gamma$ reflects the cancellation of matrix elements which occurs between the leading ($\vec{k}$-independent) terms in M_{22} and M_{31} [this term vanishes for $\epsilon_{\vec{k}} = 0$ by the second property, and it is linear in $\epsilon_{\vec{k}}/\gamma$ by the third (see also Appendix G 2)]. Terms linear in $\vec{k}$ in M_{22} and M_{31}, such as appear in Eq. (5.5), do not, in general, tend to cancel: The cancellation of matrix elements only occurs to leading order in $\vec{k}$. Thus the off-shell terms are expected to be of the same order as the on-shell terms.

Finally, let us show how the fourth property arises from the Hermiticity of the interactions on resonance. If we ignored the Hermiticity, we would estimate for a typical matrix element, using Eq. (A20) with $q_A = 0$,

$$M(\vec{k}, \vec{p}, \vec{r}, \vec{s}) \sim l_{\vec{k}}^2 l_{\vec{p}}^2 l_{\vec{r}}^2 l_{\vec{s}}^2 \Phi(\vec{k}, \vec{p}, \vec{r}, \vec{s})\Phi(\vec{r}, \vec{s}, \vec{k}, \vec{p}) \tag{5.7a}$$

$$\sim (kprs)^{-1}(rs)(kp), \tag{5.7b}$$

which is of order unity as $k \to 0$. The error in this estimate comes from setting $\Phi(\vec{k}, \vec{p}, \vec{r}, \vec{s}) \sim rs$. In fact, according to Eq. (A23),

$$\Phi(\vec{k}, \vec{p}, \vec{r}, \vec{s}) \sim \Phi(\vec{r}, \vec{s}, \vec{k}, \vec{p}) + C_0\Delta\epsilon, \tag{5.7c}$$

where C_0 is of the order of the largest momentum, which is p. On-shell ($\Delta\epsilon = 0$), the interaction is

Hermitian and the correct estimate for $\Phi(\vec{k}, \vec{p}, \vec{r}, \vec{s})$, obtained from Eq. (5.7c), yields

$$M(\vec{k}, \vec{p}, \vec{r}, \vec{s}) \sim (kprs)^{-1}(kp)^2 \sim \epsilon_{\vec{k}}/\epsilon_{\vec{s}}\,, \tag{5.7d}$$

as in Eq. (5.5a). Off shell, the incorrect estimate (5.7b) for M, when inserted into Eq. (5.6a), yields

$$\Gamma_{\vec{k}} \sim (\beta\epsilon_{\vec{k}})(1)(\tau^4\epsilon_{\vec{s}})(\epsilon_{\vec{k}}/\gamma) \tag{5.8a}$$

$$\sim \tau\epsilon_{\vec{k}}^2\,, \tag{5.8b}$$

because the small momentum s is of order γ/τ^2 [see Eq. (D16)]. Once again the correct estimate for $\Phi(\vec{k}, \vec{p}, \vec{r}, \vec{s})$, given by Eq. (5.7c), yields the smaller estimate for M:

$$M \sim (kprs)^{-1}(kp)(kp + p\gamma) \sim \gamma/\epsilon_{\vec{s}}\,, \tag{5.8c}$$

as in Eq. (5.6a). If there were a breakdown of the Hermiticity property [Eq. (5.7c)], in higher-order perturbation theory, we would presumably still have $\Gamma_{\vec{k}} \sim \epsilon_{\vec{k}}^2$ as in Eq. (5.8b), but the temperature-dependent constant of proportionality would be much larger than that found in lowest Born approximation [Eq. (5.6b)].

Thus, the four features of the second-order calculation must persist in higher orders, if the result in Eq. (5.1) is to be valid in the limit $\epsilon_{\vec{k}} \to 0$. The first two properties are demonstrated by a detailed analysis in Appendix G1, where we consider terms of arbitrary order in perturbation theory. That discussion establishes the validity of Eq. (5.3).

With regard to the other two properties, our arguments are much less complete. We shall first discuss the third property, namely, the regularity in $\epsilon_{\vec{k}}$. From an analysis of diagrams of relatively low order, we argue in Appendix G2 that there exists an n_0, such that for $\epsilon_{\vec{k}} \ll \tau^{n_0}$ the function $K(\epsilon_{\vec{k}}, \rho)$ in Eq. (5.3) may be expanded in powers of $\epsilon_{\vec{k}}/\tau^{n_0}$. From that analysis, however, it follows that the expansion is probably only asymptotic, since the higher-order terms in $\epsilon_{\vec{k}}/\tau^{n_0}$ have apparent divergences. Moreover, for diagrams with many vertices, the number n_0 appears to be quite large, although this probably reflects a weakness of our analysis, rather than a real physical effect. It would, of course, be desirable to give a more rigorous discussion of the existence of an expansion in $\epsilon_{\vec{k}}$, but such a discussion appears to be very difficult, since it must involve diagrams of arbitarily high order. In Appendix G2, this point is illustrated by considering a class of diagrams, corresponding to the interaction of a magnon with a longitudinal fluctuation of the sublattice magnetization, which leads to arbitrarily large values of n_0. In order to remove this divergence, which we believe to be unphysical, we would have to perform a complete resummation of the longitudinal spin-correlation function.[84] It is reasonable to conjecture that the interaction of a spin wave with longitudinal fluctuations is finite at long wavelengths, so that the divergence is effectively removed, but we have not demonstrated this in the present work. We shall assume that all such apparent divergences can be summed, so that for sufficiently small values of $\epsilon_{\vec{k}}$, an (asymptotic) expansion of $K(\epsilon_{\vec{k}}, \rho)$ exists, which is the third property that was required [see Eq. (5.4)]. It must be stressed here that we cannot make any firm statement concerning the value of the exponent n_0 which defines the small parameter of the expansion, $\epsilon_{\vec{k}}/\tau^{n_0}$. We believe that n_0 is a finite number, probably close to five (i.e., $\tau^{n_0} \sim \gamma$), but we cannot substantiate this conjecture in detail.

Turning to the fourth property, the absence of large temperature renormalization, we would like to determine the function $g(\rho, \tau)$ of Eq. (5.4). In particular, we shall argue that higher-order diagrams do not invalidate the result of the lowest Born approximation, namely, $g \propto \tau^3 \ln\tau f(\rho)$ [Eq. (5.1)]. Since at this point we consider it established (by properties one through three) that the decay rate is proportional to $\epsilon_{\vec{k}}^2$, it is sufficient to examine terms of lowest order in τ. In other words, we rely on Eq. (5.4) to rule out the possibility of terms of higher order in τ but of lower order in $\epsilon_{\vec{k}}$, which would dominate in the limit $k \to 0$ at fixed τ. We have thus eliminated the difficulties associated with the order in which the limits $k \to 0$ and $\tau \to 0$ are taken. In fact, using this reasoning, we argue in Appendix G3 that the family of diagrams which determines the leading τ dependence is characterized by having the minimum number of hole lines, and therefore consists of diagrams of the form shown schematically in Fig. 4. In other words, in the low-temperature limit we replace the potential coefficients $\Phi^{(i)}$, which occur in the lowest Born approximation, by effective potential coefficients, $R^{(i)}$, i.e., we use "dressed" vertices. These dressed vertices, examples of which are shown in Appendix G, can be computed with bare propagators, since the damping of internal intermediate states gives rise to terms of higher order in τ. Thus, in analogy with Eq. (4.3), we write the decay rate as

$$\Gamma(\vec{k}, \omega) = (\pi\omega_E/16S^2)(1 - e^{-\beta\hbar\omega})\int_{-\infty}^{+\infty}\varphi(\alpha)\,d\alpha$$
$$\times (2\pi)^{-6}\int d\vec{q}\, n_{\vec{p}}(1+n_{\vec{s}})(1+n_{\vec{r}})$$
$$\times [\mathfrak{M}_{22}\,\delta(\tilde{\omega} + \epsilon_{\vec{p}} - \epsilon_{\vec{r}} - \epsilon_{\vec{s}} - \alpha)$$
$$+ e^{\beta\hbar\omega}\mathfrak{M}_{31}\,\delta(-\tilde{\omega} + \epsilon_{\vec{p}} - \epsilon_{\vec{r}} - \epsilon_{\vec{s}} - \alpha)]\,, \tag{5.9}$$

where the matrix elements $\mathfrak{M}_{22}$ and $\mathfrak{M}_{31}$ are calculated from the $R^{(i)}$ in just the same way, viz., Eqs. (2.39) and (4.4), as M_{22} and M_{31} are obtained from the $\Phi^{(i)}$.[85]

It is apparent that an analysis of the decay rate necessitates a study of the matrix elements $\mathfrak{M}_{22}$ and

$\mathfrak{M}_{31}$, which in turn requires estimates of the dressed interactions, $R^{(i)}$. As we have already seen in Eq. (5.7c), in order to obtain the fourth property it is necessary to take account of the Hermiticity of the interactions on resonance. It seems likely that the dressed interactions also obey the Hermiticity relations, which take the form

$$R^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = R^{(1)}_{\vec{3}\vec{4}\vec{1}\vec{2}} \quad \text{for} \quad \epsilon_{\vec{1}} + \epsilon_{\vec{2}} = \epsilon_{\vec{3}} + \epsilon_{\vec{4}} \;, \tag{5.10a}$$

$$R^{(3)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = R^{(2)}_{\vec{3}\vec{4}\vec{1}\vec{2}} \quad \text{for} \quad \epsilon_{\vec{3}} = \epsilon_{\vec{1}} + \epsilon_{\vec{2}} + \epsilon_{\vec{4}} \;, \tag{5.10b}$$

and so forth. Although we have not been able to prove these relations to all orders in $1/z$ and $1/zS$, we believe them to be true for the following reasons: First, we have already seen that they hold in lowest Born approximation. Second, we verify in Appendix H 1 that these relations are satisfied for some third-order perturbation-theory terms. Third, they were found to hold to lowest order in $1/z$ for the anisotropic antiferromagnet.[86] That calculation is quite relevant, because in order to collect all terms of lowest order in $1/z$, it was necessary to resum completely a certain class of repeated anisotropy scatterings. Fourth, as we show in Appendix H 2, the effective interactions in a ferromagnet at low temperature satisfy this Hermiticity property. Finally, we may justify these relations by a heuristic argument, based on a hydrodynamic decoupling scheme, similar to the one used by Schwabl and Michel[87] for the ferromagnet. In that theory, the decay rate may roughly be written as

$$\Gamma_{\vec{k}} \sim \beta\omega \sum_{if} |\langle f|V|i\rangle|^2 \delta(E_f - E_i) p_i \;, \tag{5.11}$$

where p_i is the Boltzmann probability of the states i and the perturbation V is

$$V \sim [\mathcal{H}, S^-_\alpha(\vec{k})] - E_{\vec{k}} S^-_\alpha(\vec{k}) \;, \tag{5.12}$$

where $S^-_\alpha(\vec{k}) = l_{\vec{k}} S^-_a(\vec{k}) + m_{\vec{k}} S^-_b(\vec{k})$. Then one might argue that, neglecting the kinematic interactions associated with the nonorthogonality of the spin-wave states, the matrix element of V in Eq. (5.12) should be obtained from $[\mathcal{H}, \alpha^\dagger_{\vec{k}}]$, since $S^-_\alpha(\vec{k}) = 2S\alpha^\dagger_{\vec{k}}$. But $[\mathcal{H}, \alpha^\dagger_{\vec{k}}]$ corresponds to Φ_{in} [see Eq. (G8)], or in a renormalized theory to R_{in}, so that the matrix element which in our boson formalism appears as $R_{\text{in}}R_{\text{out}}$ is really $|R_{\text{in}}|^2$. Thus the Hermiticity relations (5.10), which appear in an oblique way in our theory, hold automatically, since the spin-wave interactions are themselves Hermitian. It would be highly desirable to develop a microscopic spin-wave formalism in which this property appears in a natural manner.

Let us now analyze the matrix elements $\mathfrak{M}_{22}$ and $\mathfrak{M}_{31}$ in Eq. (5.9), assuming the Hermiticity relations (5.10) for the dressed vertices. We note that the dressed vertices include a factor $\Phi^{(i)}$ at the external vertex, and, consequently, the long-wavelength estimates for the $\Phi^{(i)}$ also hold for the $R^{(i)}$. It follows that the estimates given in Eqs. (5.7d) and (5.8c) are also correct for the matrix element $\mathfrak{M}$, so that Eq. (5.6) for $\Gamma_{\vec{k}}$ remains valid. Thus we may conclude that $g \sim \tau^3 \ln\tau f(\rho)$, which together with Eq. (5.4) establishes Eq. (5.1). The function $f(\rho)$ is of order unity, and it is given in terms of the infinite series needed to construct the dressed vertices $R^{(i)}$. This series is in the parameters z^{-1} and $(zS)^{-1}$, and is quite similar to the series describing other "$1/z$ effects" such as zero-point spin deviations.[88] To illustrate this expansion, we have evaluated the dressed vertices to order $1/z$ in Appendix G 3. In conformity with our discussion, we find that the dressed vertices have the same long-wavelength behavior as the bare vertices, differing from the latter by numerical factors which are small when $zS \gg 1$. The result found in Appendix G 3 for the decay rate on resonance may be written as

$$\Gamma_{\vec{k}} = 2\omega_E S^{-2} (2\pi)^{-3} I_A \epsilon^2_{\vec{k}} F \;, \tag{5.13}$$

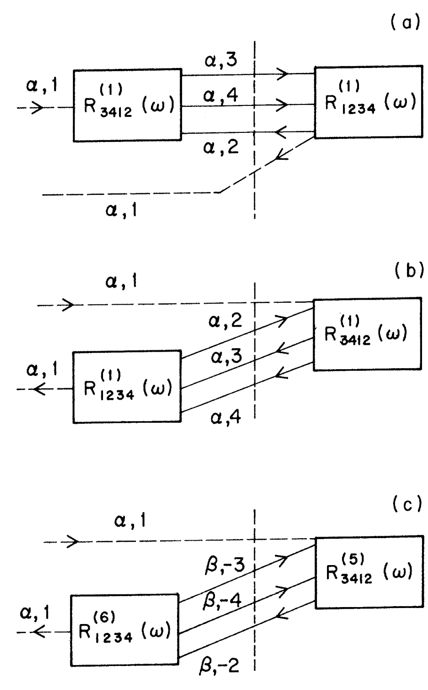

FIG. 4. Schematic form of dominant diagrams at low temperature for the imaginary part of the self-energy. Here $\vec{1}$, $\vec{2}$, etc., denote $\vec{k}_1$, $\vec{k}_2$, etc., with $\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4$. Here and below, the vertical dashed line indicates the energy-conserving δ function. Note that (a) and (b) are related by the detailed balance factor. According to the cancellation theorem the "dangerous" terms from (a) and (c) cancel one another.

with

$$F = 1 + a_1 (zS)^{-1} + \cdots \ . \tag{5.14}$$

The explicit calculation yields $a_1 = 0$, i.e., there are no vertex corrections to order $(zS)^{-1}$. This result is similar to that for the ferromagnet,[8] where only the first term in the Born series in S^{-1} contributes to the decay rate at long wavelengths and low temperatures.[89]

The results of Eq. (5.14) are not complete, in that we have not included effects due to the renormalization of the spin-wave energies. We have justified the neglect of the damping of intermediate states in detail, but from Appendix G 4, it is clear that in higher order in $1/z$ one must include self-consistently the renormalization of the real part of the spin-wave energy. Such effects can in fact be included without difficulty in our calculation of the decay rate, by suitably replacing the unperturbed spin-wave velocity by its fully renormalized value, as is done in Appendix I 1.

C. Analogy with Ferromagnet

The discussion in Sec. V B implies that the interactions between physical spin waves in an antiferromagnet vanish in the long-wavelength limit. This observation is the basis of a natural and unified picture of spin-wave interactions in Heisenberg systems. In order to develop this idea, let us first consider spin-wave interactions in a Heisenberg ferromagnet, where the magnon gas at low temperatures can be treated via a low-density expansion. Using the Dyson-Maleev representation for spin operators given in Eqs. (2.2a)–(2.2c), we obtain the decay rate as[90]

$$\Gamma(\vec{k}, \hbar^{-1}E_{\vec{k}}) \sim \beta E_{\vec{k}} \sum_{\vec{2},\vec{3},\vec{4};\vec{2}',\vec{3}',\vec{4}'} \langle \vec{k}\vec{2} | V | \vec{3}\vec{4} \rangle \langle \vec{3}'\vec{4}' | V | \vec{k}\vec{2}' \rangle \times \mathfrak{F}(\vec{k}, \vec{2}, \vec{3}, \vec{4}; \vec{k}', \vec{2}', \vec{3}', \vec{4}') \ , \tag{5.15}$$

where $\mathfrak{F}$ is a generalized scattering amplitude or vertex function. At long wavelengths, the matrix elements of the potential behave as $\langle \vec{1}\vec{2} | V | \vec{3}\vec{4} \rangle \sim \vec{k}_3 \cdot \vec{k}_4$, so that as $k \to 0$,

$$\langle \vec{k}2 | V | 34 \rangle \sim \vec{k}_3 \cdot \vec{k}_4 \sim \mathcal{O}(1) \ , \tag{5.16a}$$

whereas

$$\langle 3'4' | V | \vec{k}2' \rangle \sim \vec{k} \cdot \vec{k}_2' \sim \mathcal{O}(E_{\vec{k}}^{1/2}) \ , \tag{5.16b}$$

where we have used the fact that for a ferromagnet $E_{\vec{k}} \propto k^2$. Inserting these expressions into Eq. (5.15) and including also the next term in $\vec{k}$, we obtain

$$\Gamma(\vec{k}, \hbar^{-1}E_{\vec{k}}) \sim \beta E_{\vec{k}} \sum_{\vec{2},\vec{3},\vec{4};\vec{2}',\vec{3}',\vec{4}'} \mathfrak{F}(\vec{k}, \vec{2}, \vec{3}, \vec{4}; \vec{k}', \vec{2}', \vec{3}', \vec{4}') \times \Big[\vec{C}^{(1)} \cdot \vec{k} + \sum_{\alpha\beta} C_{\alpha\beta}^{(2)} k_\alpha k_\beta + \cdots \Big] \ , \tag{5.17}$$

where α and β are summed over the values x, y, and z. If one neglects the $\vec{k}$ dependence of $\mathfrak{F}$, one sees that by symmetry the term proportional to $\vec{k}$ in the square bracket drops out, and consequently one obtains

$$\Gamma(\vec{k}, \hbar^{-1}E_{\vec{k}}) = A E_{\vec{k}}^2 \ , \tag{5.18}$$

where A is formally given by a momentum integral. It is thus correct to view the product $\langle V \rangle \langle V \rangle$ in Eq. (5.15) as being of order $k^2 \sim E_{\vec{k}}$ rather than of order $k \sim E_{\vec{k}}^{1/2}$ as might appear from Eq. (5.16). However, since the integral A actually diverges, a detailed calculation retaining the $\vec{k}$ dependence of $\mathfrak{F}$ is necessary. Such a calculation at low temperatures, neglecting the damping in intermediate states, yields[8,28,91]

$$\Gamma(\vec{k}, \hbar^{-1}E_{\vec{k}}) \sim E_{\vec{k}}^2 [\ln(E_{\vec{k}}/k_B T)]^2 \ . \tag{5.19}$$

Perhaps a self-consistent calculation will restore the result in Eq. (5.18) at long wavelengths. In any event, the potential may still be regarded as weak for small k, since the correction to Eq. (5.18) is logarithmic.

The analogous situation holds for the antiferromagnet, where we have (see Appendix A)

$$\langle \vec{1}\vec{2} | V | \vec{3}\vec{4} \rangle \sim (E_3 E_4 / E_1 E_2)^{1/2} \ , \tag{5.20}$$

since here $E_{\vec{k}} \propto k$. In other words, for both the ferromagnet and the antiferromagnet one can write

$$\langle \vec{1}\vec{2} | V | \vec{3}\vec{4} \rangle \sim (E_1 E_2 E_3 E_4)^{1/2} / k_1 k_2 \ , \tag{5.21}$$

so that one factor in Eq. (5.15) appears to be of order $E_{\vec{k}}^{1/2}$ and the other of order $E_{\vec{k}}^{1/2}/k$. As we have discussed in Sec. V B, however, a cancellation of matrix elements occurs also in the antiferromagnet, and the product $\langle V \rangle \langle V \rangle$ appearing in Eq. (5.15) is in fact of order $E_{\vec{k}}$.

It seems clear from this discussion, that for both systems the appearance, when the damping of intermediate states is taken into account, of a "dangerous" term of order $E_{\vec{k}}/k$ in $\langle V \rangle \langle V \rangle$, is an artifact of the boson formalism, since this term cancels in the calculation of the decay rate. As mentioned earlier, this suggests that a formalism dealing directly with spin operators might avoid this term, and lead to Eq. (5.18) in a simpler way, without the necessity of a detailed cancellation.

VI. DAMPING OF SPIN WAVES IN CLASSICAL REGIME

A. Classical Formalism

In Secs. I–V, it was shown that the Born approximation gives the leading contribution to spin-wave damping at long wavelengths and low temperatures ($k_B T \ll JzS$). Moreover, this result is self-consistent and has the hydrodynamic form at long wavelengths. Since hydrodynamics is essentially a classical theory, we should expect to be able to find hydrodynamic damping in the classical regime as well, at least for some range of temperatures. As we show below, the lowest Born approximation again turns out to give the leading contribution to the spin-wave damping in the classical low-temperature regime $T \ll T_N$.

We shall obtain the classical expressions by taking the following limits:

$$\hbar \to 0, \quad J \to 0, \quad S \to \infty , \tag{6.1}$$

with

$$\hbar S = \tfrac{1}{2} N_0 \tag{6.2}$$

and

$$k_B T_0 = \mathcal{E}_E = H_E S = 2zJS^2 = \tfrac{1}{2} N_0 \omega_E \tag{6.3}$$

remaining finite. (T_0 is equal to $3T_N$ is the mean-field theory.) The classical theory for a ferromagnet has very recently been considered by Loly.[92]

The quantum-mechanical low-temperature regime considered previously is $k_B T \ll JzS \ll k_B T_0/S$ and is now vanishingly small, since $JzS \to 0$. The classical low-temperature regime is $T \ll T_0$, and since $k_B T \gg JzS$, the Bose occupation numbers are arbitrarily large:

$$n_{\vec{p}} \approx k_B T/(2JzS\epsilon_{\vec{p}}) = S(T/T_0)\epsilon_{\vec{p}}^{-1} \to \infty \tag{6.4}$$

for all values of $\vec{p}$ in the Brillouin zone. Let us therefore introduce the classical operators

$$\tilde{\alpha}_{\vec{p}} = S^{-1/2}\alpha_{\vec{p}}, \quad \tilde{\beta}_{\vec{p}} = S^{-1/2}\beta_{\vec{p}} , \tag{6.5}$$

with finite occupation numbers $\tilde{n}_{\vec{p}} = S^{-1} n_{\vec{p}}$. The Hamiltonian is then

$$\mathcal{H} = \mathcal{E}_E \sum_{\vec{p}} \epsilon_{\vec{p}} (\tilde{\alpha}_{\vec{p}}^\dagger \tilde{\alpha}_{\vec{p}} + \tilde{\beta}_{\vec{p}}^\dagger \tilde{\beta}_{\vec{p}}) - (4N)^{-1} \sum_{\vec{1}\vec{2}\vec{3}\vec{4}} \delta_{\vec{k}}(\vec{1}+\vec{2}-\vec{3}-\vec{4})\, l_{\vec{1}} l_{\vec{2}} l_{\vec{3}} l_{\vec{4}} (\Phi^{(1)} \tilde{\alpha}_{\vec{1}}^\dagger \tilde{\alpha}_{\vec{2}}^\dagger \tilde{\alpha}_{\vec{3}} \tilde{\alpha}_{\vec{4}} + \cdots) , \tag{6.6}$$

where the dots denote the other terms in the Hamiltonian, which are the same as in Eq. (2.17) with α and β replaced by $\tilde{\alpha}$ and $\tilde{\beta}$, and $\epsilon_{\vec{p}}$, l_i, and $\Phi^{(i)}$ are dimensionless functions of the $\vec{k}_i$. The interaction term is of relative order $\tilde{\alpha}_{\vec{p}}^\dagger \tilde{\alpha}_{\vec{p}} \approx (T/T_0)\epsilon_{\vec{p}}^{-1}$ compared to the quadratic term. This means that each interaction leads to an additional power of the small parameter T/T_0, at low temperatures. Since the occupation numbers no longer limit the integrals to small intermediate momenta, however, we must redo the Born-approximation calculation of Sec. III in this case.

B. Born Approximation On-Resonance

In the classical regime, Eq. (2.38) for the decay rate may be written as

$$\Gamma_{\vec{k}} = \tfrac{1}{16}\pi\omega_E (T/T_0)^2 \epsilon_{\vec{k}}^2 L , \tag{6.7}$$

where

$$L = N^{-2} \sum_{\vec{p},\vec{s}} (\epsilon_{\vec{p}}\epsilon_{\vec{r}}\epsilon_{\vec{s}}\epsilon_{\vec{k}})^{-1} \delta(\epsilon_{\vec{k}} + \epsilon_{\vec{p}} - \epsilon_{\vec{r}} - \epsilon_{\vec{s}}) \times M_{22}(\vec{k}, \vec{p}, \vec{r}, \vec{s}) , \tag{6.8}$$

with M_{22} given in Eq. (2.39). In obtaining this result we have taken the limits described in Eq. (6.1). As mentioned above, the momentum integrals are not restricted by the occupation numbers, but only by the energy-conserving δ function and the condition $\epsilon_{\vec{k}} \ll 1$. As shown in Appendix E, for any $q \neq 0$, we have $\epsilon_{\vec{p}} < \epsilon_{\vec{q}} + \epsilon_{\vec{p}-\vec{q}}$. It follows that for $\epsilon_{\vec{k}} \ll 1$ and $s \leq \frac{1}{2}|\vec{p}-\vec{k}|$, the δ function is only satisfied for $s \approx 0$, or equivalently for $\vec{r} \approx \vec{p}$. For this region, we may use the small-k and small-s expansions of the Φ coefficients given in Appendix A. The momentum p, on the other hand, ranges over the entire Brillouin zone. For $s > \frac{1}{2}|\vec{p}-\vec{k}|$, the contribution to Eq. (6.8) is obtained from the case $s < \frac{1}{2}|\vec{p}-\vec{k}|$ by interchanging $\vec{s}$ and $\vec{r}$ as in the quantum case. From the expressions for M_{22}, l_i [Eqs. (2.13) and (2.14)], and the limiting forms given in Eqs. (A14) and (A17), we find, in the limit $\epsilon_{\vec{k}} \ll 1$, $\epsilon_{\vec{s}} \ll 1$, $\epsilon_{\vec{k}}/\epsilon_{\vec{p}} \ll 1$, and $\epsilon_{\vec{s}}/\epsilon_{\vec{p}} \ll 1$,

$$\tfrac{1}{4} M_{22} = (\epsilon_{\vec{k}}/\epsilon_{\vec{s}})[1 - \vec{v}_{\vec{p}} \cdot \hat{k} - (\Delta + \epsilon_{\vec{k}})/\epsilon_{\vec{k}}]\,(1 - \vec{v}_{\vec{p}} \cdot \hat{k}) , \tag{6.9}$$

where

$$\Delta = \epsilon_{\vec{p}} - \epsilon_{\vec{s}} - \epsilon_{\vec{r}} = \tfrac{1}{2}[-s + \vec{v}_{\vec{p}} \cdot (\vec{s} - \vec{k})] . \tag{6.10}$$

Inserting this expression into Eq. (6.8) and multiplying by 2 to take into account the domain, $s > \frac{1}{2}|\vec{k}-\vec{p}|$ (i.e., $r \approx 0$), we find

$$L = 16N^{-2} \sum_{\vec{p},\vec{s};\, s\approx 0} (\epsilon_{\vec{p}}\epsilon_{\vec{s}})^{-2} \delta(k - s + \vec{v}_{\vec{p}} \cdot (\vec{s} - \vec{k})) \times (1 - \hat{k} \cdot \vec{v}_{\vec{p}})^2 . \tag{6.11}$$

This integral has the same form as the one for the quantum case, cf. Eq. (3.21). We may replace the sum over $\vec{s}$ by an integral and repeat the steps leading to Eq. (3.27), with minor modifications, to find

$$L = 16\pi^{-2} \textstyle\sum_{\vec{p}} \epsilon_{\vec{p}}^{-2} v_{\vec{p}}^{-1} (1 - v_{\vec{p}}\mu)^2 \ln[(1 + v_{\vec{p}})/(1 - v_{\vec{p}})] , \tag{6.12}$$

with $\mu = \hat{v}_{\vec{p}} \cdot \hat{k}$ and $v_{\vec{p}} = |\vec{v}_{\vec{p}}|$. As before, we perform the sum over μ in Eq. (6.12) by replacing $(1 - v_{\vec{p}}\mu)^2$ by its average under the symmetry operations, which for cubic symmetry is

$$\langle (1 - v_{\vec{p}}\mu)^2 \rangle = 1 + \tfrac{1}{3} v_{\vec{p}}^2 , \tag{6.13}$$

independent of the direction of $\vec{k}$. Equations (6.7), (6.12), and (6.13) then lead to

$$\Gamma_{\vec{k}} = (4\eta/3\pi)\omega_E (T/T_0)^2 \epsilon_{\vec{k}}^2 , \tag{6.14}$$

where the numerical constant η is given by

$$\eta = N^{-1} \sum_{\vec{p}} \epsilon_{\vec{p}}^{-2} v_{\vec{p}}^{-1} (\tfrac{3}{4} + \tfrac{1}{4} v_{\vec{p}}^2) \ln[(1+v_{\vec{p}})/(1-v_{\vec{p}})] , \tag{6.15}$$

and the sum goes over the entire zone. At the zone edge, where $v_{\vec{p}} \to 0$, the integrand remains finite. We have not acutally carried out the sum in (6.15), but we estimate the value of η to be roughly 2.

C. Born Approximation Off-Resonance and Self-Consistency

For $\tilde{\omega} \neq \epsilon_{\vec{k}}$, we must again take into account the term involving M_{31}, which in the limit $\epsilon_{\vec{s}}/\epsilon_{\vec{p}} \ll 1$, $\epsilon_{\vec{k}}/\epsilon_{\vec{p}} \ll 1$ takes the form

$$\tfrac{1}{4} M_{31} = -(\epsilon_{\vec{k}}/\epsilon_{\vec{s}})(1+\vec{v}_{\vec{p}}\cdot\hat{k})[1-\vec{v}_{\vec{p}}\cdot\hat{k}-(\epsilon_{\vec{k}}+\Delta)/\epsilon_{\vec{k}}] . \tag{6.16}$$

Then the decay rate is

$$\Gamma(\vec{k},\omega) = \tfrac{1}{16}\pi\omega_E (T/T_0)^2 \tilde{\omega}\epsilon_{\vec{k}} \mathcal{L}(\epsilon_{\vec{k}},\rho) , \tag{6.17}$$

with

$$\mathcal{L}(\epsilon_{\vec{k}},\rho) = 2N^{-2} \sum_{\vec{p}\vec{s};\, s\approx 0} (\epsilon_{\vec{p}}^2 \epsilon_{\vec{s}} \epsilon_{\vec{k}})^{-1} \times [\delta(\epsilon_{\vec{k}}\rho+\Delta) M_{22} + \delta(\epsilon_{\vec{k}}\rho-\Delta) M_{31}] . \tag{6.18}$$

Once again, this expression may be evaluated by repeating the steps which led to Eq. (4.11) in the quantum case, and the answer is

$$\Gamma(\vec{k},\omega) = (4/3\pi)\omega_E (T/T_0)^2 \tilde{\omega}\epsilon_{\vec{k}} (\tfrac{1}{4}\eta_2 + \tfrac{3}{4}\eta_0 \rho) , \tag{6.19}$$

where again $\rho = \tilde{\omega}/\epsilon_{\vec{k}}$, $\tilde{\omega} = \omega/\omega_E$, $\epsilon_{\vec{k}} = \frac{1}{2}k$, and

$$\eta_l = N^{-1} \sum_{\vec{p}} \epsilon_{\vec{p}}^{-2} v_{\vec{p}}^{l-1} \ln[(1+v_{\vec{p}})/(1-v_{\vec{p}})] . \tag{6.20}$$

Note that $\frac{1}{4}\eta_2 + \frac{3}{4}\eta_0 = \eta$, so that on-resonance ($\rho = 1$) we recover our previous results.

The self-consistency check proceeds as in the quantum case, the result being unchanged when the finite decay rate, $\Gamma_{\rm th} \approx \omega_E (T/T_0)^2$, of intermediate magnons is taken into account. Similarly, the discussion of higher-order terms is analogous to that given for the quantum regime in Sec. V. It is expected that these terms will be smaller than those in Eq. (6.19) by a factor of order T/T_0 or higher.

The formulas in this section were written in the classical limit, Eq. (6.1), where the spin S tends to infinity and drops out of the formalism. We may clearly keep S finite but large, and use these expressions in the range $2JzS \ll k_B T \ll 2JzS^2$, i.e., $S^{-1}T_N \ll T \ll T_N$. There will then be corrections not only of relative order k and T/T_N, but also of relative order $T_N/ST \approx 2JzS/k_B T$.

VII. SPIN-CORRELATION FUNCTIONS AND COMPARISON WITH HYDRODYNAMICS

Having obtained the damping rates for both the quantum and classical cases, we may now calculate the spin Green's functions $\mathcal{G}_Q^{+-}$ and $\mathcal{G}_S^{+-}$ given in Eq. (2.36). In Sec. VII A we construct simple approximate forms for the spin spectral-weight functions valid at low temperatures in the long-wavelength limit. From these functions we obtain the correlation functions C_Q^{+-} and C_S^{+-} for the staggered and total spin, respectively. These correlation functions are then compared in Sec. VII B with the hydrodynamic results. In this way we not only verify the hydrodynamic predictions, but we also obtain an evaluation, to lowest order in the temperature, of the thermodynamic and transport coefficients introduced phenomenologically in the hydrodynamic theory.[13] Finally, in Sec. VII C we compare our results with those obtained for phonon systems.

A. Spin Spectral-Weight Function

In order to simplify the algebra, we shall obtain representations of the spin Green's functions at low temperatures which are valid to lowest order in the small parameters $\epsilon_{\vec{k}}$, τ, and $1/zS$. (In Appendix I, the treatment of this section is generalized to include terms of arbitrary order in $1/z$ and $1/zS$.) To do this we start from Eq. (2.33) which relates the "spin normal-mode" Green's functions $\mathcal{G}_{\mu\nu}$ to their boson counterparts $G_{\mu\nu}$:

$$\underline{\mathcal{G}} = (\underline{1} + \underline{\Lambda})\underline{G} \tag{7.1}$$

in matrix notation. We wish to construct a simple expression for the spin spectral-weight function,[93] $\mathrm{Im}\mathcal{G}_{\mu\nu}(\vec{k},\omega)$. In deciding which terms to keep it is helpful to use the results of the analysis of Sec. V and Appendix G, where we justified the following asymptotic momentum dependences:

$$\mathrm{Re}\underline{\Lambda} \sim \mathcal{O}(1) , \tag{7.2a}$$

$$\mathrm{Im}\underline{\Lambda} \sim \mathcal{O}(\epsilon_{\vec{k}}) , \tag{7.2b}$$

$$\mathrm{Re}\underline{\Sigma} \sim \mathcal{O}(\epsilon_{\vec{k}}) , \tag{7.2c}$$

$$\mathrm{Im}\underline{\Sigma} \sim \mathcal{O}(\epsilon_{\vec{k}}^2) . \tag{7.2d}$$

Furthermore, since $\underline{\Lambda}$ and $\underline{\Sigma}$ involve the interaction between spin waves, we may estimate the numerical coefficients in Eq. (7.2), for $\tau \ll 1$, $(zS)^{-1} \ll 1$, as

$$\mathrm{Re}\underline{\Lambda} \ll 1 , \tag{7.3a}$$

$$\mathrm{Im}\underline{\Lambda}/\epsilon_{\vec{k}} \ll 1 , \tag{7.3b}$$

$$\mathrm{Re}\underline{\Sigma}/E_{\vec{k}} \ll 1 , \tag{7.3c}$$

$$\omega_E \mathrm{Im}\underline{\Sigma}/E_{\vec{k}}^2 \ll 1 . \tag{7.3d}$$

Note that we have already assumed $\|\underline{\Sigma}\| \ll E_{\vec{k}}$ to obtain the results of Eq. (2.26). Use of Eq. (7.3a) also allows us to neglect the term $(\mathrm{Re}\Lambda)(\mathrm{Im}\underline{G})$ in comparison to $\mathrm{Im}\underline{G}$ in Eq. (7.1) so that

$$\mathrm{Im}\underline{\mathcal{G}} \approx \mathrm{Im}\underline{G} + (\mathrm{Im}\underline{\Lambda})(\mathrm{Re}\underline{G}) . \tag{7.4}$$

To evaluate the first term in Eq. (7.4) it is convenient to rewrite Eq. (2.26) as

$$\underline{G} = \underline{G}_D + \underline{G}_D \underline{\Sigma}_O \underline{G}_D , \tag{7.5}$$

where the subscripts D and O indicate the diagonal and off-diagonal parts of the matrices, respectively. In terms of these matrices, one can write[67]

$$\mathrm{Im}\underline{G}_D = \underline{G}_d^2 \underline{\Sigma}_D'' , \tag{7.6}$$

where $\underline{G}_d$ is a diagonal matrix with elements $\sin(\rho - 1)|G_{\alpha\alpha}(\vec{k}, \omega - i0^+)|$ and $\mathrm{sgn}(-1-\rho) \times |G_{\beta\beta}(\vec{k}, \omega - i0^+)|$, where $\mathrm{sgn}(x) = x/|x|$. Note that the sgn factors do not appear in Eq. (7.6), since there only $\underline{G}_d^2$ is involved. We include these phase factors so that $\underline{G}_d$ can be used below as an approximation to $\mathrm{Re}\underline{G}_D$. Now let us consider the second term on the right-hand side of Eq. (7.5). On resonance, i.e., for $|\omega - \epsilon_{\vec{k}}| \sim ||\underline{\Sigma}||$, this term is dominated by the term $\underline{G}_D$. Accordingly, we need to develop an expression for $\underline{G}_D \underline{\Sigma}_O \underline{G}_D$ which is valid off resonance. In that case, using Eq. (7.3) we see that to lowest order in τ and $(zS)^{-1}$ we can neglect higher powers of $\underline{\Sigma}$, so that

$$\mathrm{Im}\underline{G}_D \underline{\Sigma}_O \underline{G}_D \approx \underline{G}_d \underline{\Sigma}_0'' \underline{G}_d . \tag{7.7}$$

Combining Eqs. (7.6) and (7.7), we may write the boson Green's function as

$$\mathrm{Im}\underline{G} = \underline{G}_d \underline{\Sigma}'' \underline{G}_d . \tag{7.8}$$

Next let us consider the second term in Eq. (7.4). Here for $\underline{G}$ we need only use the first term in Eq. (7.5), since terms of order $\underline{\Sigma}\,\underline{\Lambda}$ may be dropped. Accordingly, we have

$$(\mathrm{Im}\underline{\Lambda})(\mathrm{Re}\underline{G}) \approx (\mathrm{Im}\underline{\Lambda})(\mathrm{Re}\underline{G}_D) . \tag{7.9}$$

Furthermore, it is easily seen that this term is negligible on resonance. Therefore, it is permissible to replace $\mathrm{Re}\underline{G}_D$ by $\underline{G}_d$. Moreover, off resonance $\underline{G}_d$ is well approximated by the unperturbed Green's function, $\underline{G}^0$, so that

$$(\mathrm{Im}\underline{\Lambda})(\mathrm{Re}\underline{G}) \approx \underline{G}_d (\underline{G}^0)^{-1} (\mathrm{Im}\underline{\Lambda}) \underline{G}_d . \tag{7.10}$$

Finally, combining this result with Eq. (7.8) we write Eq. (7.4) in the form

$$\mathrm{Im}\underline{\mathcal{G}} = \underline{G}_d \,\mathrm{Im}\underline{\Sigma}^s \underline{G}_d , \tag{7.11}$$

with

$$\underline{\Sigma}^S = \underline{\Sigma} + (\underline{G}^0)^{-1}\underline{\Lambda} . \tag{7.12}$$

Thus we see that $\underline{\Sigma}^S$ plays the role of a mass operator for the "spin normal-mode" Green's function $\mathcal{G}_{\mu\nu}(\vec{k}, \omega)$. That is, $\mathcal{G}_{\mu\nu}(\vec{k}, \omega)$ bears the same relation to $\Sigma^S_{\mu\nu}(\vec{k}, \omega)$, viz., Eq. (2.26), as its boson counterparts do to the boson self-energies.[94]

Let us now explicitly construct $\mathrm{Im}\Sigma^S_{\mu\nu}(\vec{k}, \omega)$. We write the results of Eqs. (4.24) and (6.19) as

$$\hbar^{-1}\Sigma''_{\alpha\alpha}(\vec{k}, \omega) = \Gamma(\vec{k}, \omega) = \omega_E \epsilon_{\vec{k}}^2 (C_1^0 \rho + C_2^0 \rho^2) , \tag{7.13}$$

where

$$C_1^0 = \tfrac{1}{2} S^{-2} (2\pi)^{-3} I_A \ll 1 , \tag{7.14a}$$

$$C_2^0 = 3 C_1^0 \ll 1 , \tag{7.14b}$$

in the quantum case, and

$$C_1^0 = (\eta_2/3\pi)(T/T_0)^2 \ll 1 , \tag{7.15a}$$

$$C_2^0 = (\eta_0/\pi)(T/T_0)^2 \ll 1 , \tag{7.15b}$$

in the classical case. Furthermore, as shown in Appendix B, we may write in both the classical and quantum low-temperature limits

$$\Sigma''_{\alpha\alpha}(\vec{k}, \omega) = \Sigma''_{\beta\beta}(\vec{k}, -\omega) = \Sigma''_{\alpha\beta}(\vec{k}, -\omega) = -\Sigma''_{\beta\alpha}(\vec{k}, \omega) . \tag{7.16}$$

In addition, we shall need the evaluations of $\Lambda''_{\mu\nu}(\vec{k}, \omega)$ from Appendix C. We write the results for both the quantum case, Eq. (C24), and the classical case, Eq. (C29), in the form

$$\Lambda''_{\mu\nu}(\vec{k}, \omega) = -C_2^0 \epsilon_{\vec{k}} \rho . \tag{7.17}$$

Inserting these evaluations into Eq. (7.12) we obtain the spin normal-mode mass operator as

$$\hbar^{-1}\mathrm{Im}\Sigma^S_{\beta\beta}(k, \omega) = \hbar^{-1}\mathrm{Im}\Sigma^S_{\alpha\alpha}(k, \omega) = (C_1^0 + C_2^0)\,\omega_E \epsilon_{\vec{k}}^2 \rho , \tag{7.18a}$$

$$\hbar^{-1}\mathrm{Im}\Sigma^S_{\beta\alpha}(k, \omega) = \hbar^{-1}\mathrm{Im}\Sigma^S_{\alpha\beta}(k, \omega) = (C_2^0 - C_1^0)\,\omega_E \epsilon_{\vec{k}}^2 \rho . \tag{7.18b}$$

Note that in contrast to $\underline{\Sigma}$, the physically interesting mass operator $\underline{\Sigma}^S$ does satisfy the stability criterion that $\mathrm{Im}\,\omega\Sigma^S_{\mu\nu}(\omega)$ be positive definite. In view of Eq. (7.11) this evaluation of $\mathrm{Im}\underline{\Sigma}^S$ yields directly an evaluation of the spin spectral-weight function $\mathrm{Im}\underline{\mathcal{G}}$.

B. Comparison of Spin-Correlation Functions with Hydrodynamics

In this subsection we shall construct the total and staggered spin Green's functions defined in Eq. (2.36). Taking $(l_{\vec{k}} - m_{\vec{k}})^2 = 2/\epsilon_{\vec{k}}$ and using Eq. (7.18) for $\mathrm{Im}\underline{\Sigma}^S$ we find[95]

$$\mathrm{Im}\,\mathcal{G}_Q^{+-} = \left(\frac{4S\rho}{H_E\,\epsilon_{\vec{k}}}\right)\left(\frac{C_1^0 + C_2^0}{(\rho-1)^2 + (C_1^0 + C_2^0)^2 \epsilon_{\vec{k}}^2} + \frac{C_1^0 + C_2^0}{(\rho+1)^2 + (C_1^0 + C_2^0)^2 \epsilon_{\vec{k}}^2}\right.$$

$$-\frac{2(C_2^0-C_1^0)\,\mathrm{sgn}(1-\rho^2)}{|(\rho-1)+i(C_1^0+C_2^0)\epsilon_{\vec{k}}|\,|(\rho+1)+i(C_1^0+C_2^0)\epsilon_{\vec{k}}|}\Bigg)\,, \tag{7.19}$$

which we write to lowest order in C_1^0 and C_2^0 as

$$\mathrm{Im}\mathcal{G}_Q^{+-}=\frac{(16S\rho/H_E\,\epsilon_{\vec{k}})(C_1^0+C_2^0\rho^2)}{[(\rho-1)^2+(C_1^0+C_2^0)^2\epsilon_{\vec{k}}^2]\;[(\rho+1)^2+(C_1^0+C_2^0)^2\epsilon_{\vec{k}}^2]}\,. \tag{7.20a}$$

A similar calculation gives the total spin Green's function as

$$\mathrm{Im}\mathcal{G}_S^{+-}=\frac{[4S\rho\epsilon_{\vec{k}}/H_E][C_2^0+C_1^0\rho^2]}{[(\rho-1)^2+(C_1^0+C_2^0)^2\epsilon_{\vec{k}}^2][(\rho+1)^2+(C_1^0+C_2^0)^2\epsilon_{\vec{k}}^2]}\,. \tag{7.20b}$$

These functions are also the Fourier transforms of response functions, and as such can be related to Fourier transforms of correlation functions. We define [see Eq. (2.34)]

$$C_Q^{+-}(\vec{k},\,t)=\tfrac{1}{2}\langle\{\hbar Q_{\vec{k}}^{+}(t),\;\hbar Q_{\vec{k}}^{-}(0)\}\rangle\,, \tag{7.21a}$$

$$C_S^{+-}(\vec{k},\,t)=\tfrac{1}{2}\langle\{\hbar S_{\vec{k}}^{+}(t),\;\hbar S_{\vec{k}}^{-}(0)\}\rangle\,, \tag{7.21b}$$

where the curly bracket is an anticommutator. The Fourier transforms of these correlation functions at low frequency ($\hbar\omega\ll k_BT$) are proportional to the spin Green's functions:

$$C_Q^{+-}(\vec{k},\,\omega)=\hbar^2(2k_BT/\omega)\,\mathrm{Im}\mathcal{G}_Q^{+-}(\vec{k},\,\omega-i0^+)\,, \tag{7.22a}$$

$$C_S^{+-}(\vec{k},\,\omega)=\hbar^2(2k_BT/\omega)\,\mathrm{Im}\mathcal{G}_S^{+-}(\vec{k},\,\omega-i0^+)\,. \tag{7.22b}$$

These relations yield the results

$$C_Q^{+-}(\vec{k},\,\omega)=\left(\frac{32S\hbar k_BT}{\omega_E^2\epsilon_{\vec{k}}^2}\right)\frac{C_1^0+C_2^0\rho^2}{[(\rho-1)^2+(C_1^0+C_2^0)^2\epsilon_{\vec{k}}^2][(\rho+1)^2+(C_1^0+C_2^0)^2\epsilon_{\vec{k}}^2]}\,, \tag{7.23a}$$

$$C_S^{+-}(\vec{k},\,\omega)=\left(\frac{8S\hbar k_BT}{\omega_E^2}\right)\frac{C_2^0+C_1^0\rho^2}{[(\rho-1)^2+(C_1^0+C_2^0)^2\epsilon_{\vec{k}}^2][(\rho+1)^2+(C_1^0+C_2^0)^2\epsilon_{\vec{k}}^2]}\,. \tag{7.23b}$$

Let us rewrite these functions in terms of the variables $k=2\epsilon_{\vec{k}}$, $\omega=\omega_E\epsilon_{\vec{k}}\rho$, and $c=\frac{1}{2}\omega_E$:

$$C_Q^{+-}(\vec{k},\,\omega)=\left(\frac{32k_BTS\hbar}{ck^2}\right)\frac{D'c^2k^4+cC_2^0k^2(\omega^2-c^2k^2)}{[(\omega-ck)^2+(\frac{1}{2}D'k^2)^2][(\omega+ck)^2+(\frac{1}{2}D'k^2)^2]}\,, \tag{7.24a}$$

$$C_S^{+-}(\vec{k},\,\omega)=\left(\frac{2k_BTS\hbar}{c}\right)\frac{D'c^2k^4+cC_1^0k^2(\omega^2-c^2k^2)}{[(\omega-ck)^2+(\frac{1}{2}D'k^2)^2][(\omega+ck)^2+(\frac{1}{2}D'k^2)^2]}\,, \tag{7.24b}$$

where

$$D'=\tfrac{1}{2}(C_1^0+C_2^0)\omega_E=c(C_1^0+C_2^0)\,. \tag{7.25}$$

From the hydrodynamic theory we have the results [Eq. (6.11) of Ref. 13]

$$C_Q^{+-}(\vec{k},\,\omega)=2C_{n_y,n_y}=\left(\frac{4N_0^2k_BT}{\rho_Sk^2}\right)\frac{Dc^2k^4+\rho_S\zeta k^2(\omega^2-c^2k^2)}{[(\omega-ck)^2+(\frac{1}{2}Dk^2)^2][(\omega+ck)^2+(\frac{1}{2}Dk^2)^2]}\,, \tag{7.26a}$$

$$C_S^{+-}(\vec{k},\,\omega)=2C_{m_y,m_y}=(4\chi k_BT)\frac{Dc^2k^4+\chi^{-1}K_\perp k^2(\omega^2-c^2k^2)}{[(\omega-ck)^2+(\frac{1}{2}Dk^2)^2][(\omega+ck)^2+(\frac{1}{2}Dk^2)^2]}\,, \tag{7.26b}$$

where

$$N_0=\hbar\langle Q_0^z\rangle=2S\hbar\,, \tag{7.27a}$$

$$\chi=\rho_s/c^2\,, \tag{7.27b}$$

and ζ and $K_\perp$ are transport coefficients analogous to second viscosities in superfluid helium,[15] which are related to D by

$$D=\chi^{-1}K_\perp+\rho_s\zeta\,. \tag{7.28}$$

Comparing Eqs. (7.24) and (7.26), we see that the microscopic calculations do yield the hydrodynamic form, and hence we may make the identifications

$$\rho_s = \tfrac{1}{2}\hbar c S \ , \tag{7.29a}$$

$$D = D' = \tfrac{1}{2}\omega_E (C_1^0 + C_2^0) \ , \tag{7.29b}$$

$$\rho_s \zeta = \tfrac{1}{2}\omega_E C_2^0 \ , \tag{7.29c}$$

$$\chi^{-1} K_\perp = \tfrac{1}{2}\omega_E C_1^0 \ . \tag{7.29d}$$

We also verify that the hydrodynamic prediction equation (7.28) is fulfilled.

The present derivation, valid in the low-temperature quantum ($\tau \ll 1$) or classical ($T/T_N \ll 1$) regimes, represents a microscopic calculation of the hydrodynamic parameters correct to lowest order in $1/z$. In Appendix I1, the calculation of the spin-correlation functions is generalized to include effects of arbitrary order in $1/z$ at low temperatures. Although the forms of the vertex functions $\underline{\Sigma}$ and $\underline{\Lambda}$ at long wavelengths are known, the evaluation of the coefficients appears to require an essentially intractable resummation in $1/z$.[88] Therefore, in Appendix I1 we express the spin-correlation functions in terms of these unknown numerical constants, which are then analogous to the Landau parameters in the theory of a Fermi liquid.[96] From this more extensive treatment we find that the spin-correlation functions are of the hydrodynamic form to all orders in $1/z$ and $1/zS$, providing certain relations between the vertex functions are satisfied. We verify in Appendix I2 that these relations hold in low-order perturbation theory. Presumably, they are quite general and possibly follow directly from the conservation laws, in which case they would be analogous to the Ward identities developed in the microscopic derivation of Fermi liquid theory.[96,63]

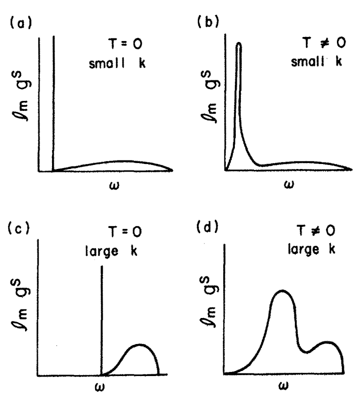

FIG. 5. Schematic diagram of $\mathrm{Im}\,\mathcal{G}_S^{+-}(\vec{k}, \omega)$, the spectral weight of the total spin-correlation function. (The staggered spin function is qualitatively similar.) Only the positive frequency domain is shown. At zero temperature [(a) and (c)], the spectral-weight function consists of a sharp single-particle excitation and, above this, a diffuse multiple-magnon band whose presence is a manifestation of zero-point motion. At finite temperatures [(b) and (d)], the single-particle excitation develops a width, although the width remains small at long wavelengths. The relative intensity of the multiple-magnon band is of order $(zS)^{-1}$ for short wavelengths at low temperatures, and its weight goes to zero in the limit $k \to 0$.

It is perhaps worthwhile to make a few general remarks about the spin-wave spectrum and the spin-correlation functions outside the hydrodynamic regime considered here. In particular, one might ask whether spin-wave interactions and zero-point motion will lead to finite lifetimes for the elementary excitations at any wavelength, for $T = 0$.[97] In the case of liquid helium, it is well known that, because of the shape of the energy spectrum, the requirement of energy and momentum conservation forbids the spontaneous decay of one excitation into two or more excitations.[15] As a result, the elementary excitations have infinite lifetimes at zero temperature. As shown in Appendix E, the same property holds for the unperturbed magnon energy spectrum, and in view of the results of the previous sections, it probably also holds self-consistently for the true renormalized spectrum. Thus the dressed magnons have zero width at $T = 0$ for all wavelengths. This means that the spin spectral-weight function has a δ-function contribution at the magnon energy. At $T = 0$, the effects of spin-wave interactions and zero-point motion are manifested in the spin spectral-weight function by the presence of multiple-magnon bands at energies above the single-magnon peak. These bands correspond to processes in which a single magnon with frequency above the resonance frequency decays, either directly or indirectly via higher-order processes, into several lower-momentum magnons. In general, the weight in an n-magnon band varies as some power of z^{-1} and will vanish at long wavelengths. The weight in these bands is thus a reflection of "depletion" or "renormalization" effects similar to those present in Bose[19,27] and Fermi[63,96] liquids. To illustrate this discussion the spin spectral-weight function is shown schematically in Fig. 5. Note that the single-magnon peak, which is a δ function at $T = 0$, broadens progressively as the temperature is raised. Above T_N the spin waves disappear completely at long wavelengths, but may still exist for shorter wavelengths.[98]

C. Physical Conditions for Hydrodynamics

The lowest Born-approximation calculations for $\Sigma_{\mu\nu}(\vec{k}, \omega)$ and $\Lambda_{\mu\nu}(\vec{k}, \omega)$ lead to the hydrodynamic forms (7.13) for the correlation functions, in the long-wavelength limit defined by

$$\omega \ll \omega_E \tau^3 \ln \tau = \omega_{tr} \quad \text{(quantum mechanically)} \ , \tag{7.30a}$$

$$\omega \ll \omega_E = \omega_{tr} \qquad \text{(classically)}\ . \qquad (7.30b)$$

As was mentioned in the Introduction, the limiting frequency for hydrodynamics ω_{tr}, defined in Eq. (7.30), is much larger than the width of the thermal magnons Γ_{th}, which is of order $\omega_E S^{-2}\tau^5$ and $\omega_E(T/T_0)^2$, in the quantum and classical cases, respectively. This situation is different from the case of first sound in helium[31–33] or in a crystal,[31,36–38] where even in the lowest approximation, the breakdown of hydrodynamics occurs when the frequency of the incident phonon reaches Γ_{th}.

Some insight into the reason for this difference can be gained by considering the lowest Born calculation of $\Sigma''_{\mu\nu}(\vec{k}, \omega)$ for finite ω, but for $k \to 0$. For the phonon system, this is equivalent to the calculation of the energy dissipation in the presence of a hypothetical force causing the system to expand and contract uniformly, at a frequency ω. The anharmonic force responsible for decay of a sound wave comes principally from terms in the Hamiltonian that are *cubic* in the phonon amplitudes. These terms give rise to a shift in the phonon frequencies linear in the dilation of the medium. Because of collisions among thermal phonons these will relax towards the thermal equilibrium appropriate to their shifted energies in a time Γ_{th}^{-1}. If ω is small compared to Γ_{th}, there will be a small hysteresis loss in this process, with a dissipation rate proportional to ω^2. For ω much larger than Γ_{th}, however, the distribution of thermal phonons does not have time to change at all during the contraction-dilation cycles, and the dissipation will be smaller than the ω^2 extrapolation from low frequencies.

From the microscopic point of view, the foregoing argument corresponds to a calculation of the transition rate for a process in which a thermal phonon is scattered, with the emission or absorption of an off-resonance phonon of wave vector zero and frequency ω. Since the momentum transferred to the thermal phonon is zero, it is clear that this process cannot proceed if the frequency ω is much larger than the spectral width Γ_{th} of the thermal phonon. More generally, for finite k, but for ω/Γ_{th} large, the scattering rate will depend in a complicated way on the ratio of k to ω and will depend on the deviation from linearity of the spectrum of thermal phonons, etc. Nonetheless, consideration of the $k=0$ limit is sufficient to show that hydrodynamics cannot be valid for $\omega/\Gamma_{th} \gg 1$ in the phonon case.

For the isotropic antiferromagnet there are no terms in the Hamiltonian that are cubic in the spin-wave amplitudes. This may be seen by considering a long-wavelength spin wave as a gradient in the direction of alignment of the staggered magnetization and/or a small deviation of the total magnetization from its equilibrium value of zero.[13] It is clear by symmetry that there can be no shift in the frequencies of the local thermal magnons linear in the amplitudes of such a spin wave.[99] Thus the hysteresis process responsible for the energy loss in the case of sound waves does not have a direct analog in the case of spin waves. The *quartic* term in the Hamiltonian does not require collisions of the thermal magnons for the effective restoration of equilibrium. Consider, for example, the calculation of $\Sigma''_{\mu\nu}(\vec{k}, \omega)$ in regime A performed in Sec. IV, where the wave vector is equal to zero. The process where a thermal magnon collides with the incident, off-resonance magnon with zero wave vector and nonzero frequency ω, can occur even when Γ_{th} is taken to zero, since this process produces *two* short-wavelength magnons which can take up the necessary momentum and energy. Further, we have seen that when the contributions from the various diagrams corresponding to magnon emission and absorption are added together, the total decay rate does not depend on the ratio of ω to Γ_{th}. Thus the condition $\omega/\Gamma_{th} \ll 1$ is not necessary for for establishing local equilibrium in the antiferromagnet. The actual conditions that we have found in Eq. (7.30) depend on the details of the scattering process, and are different for the classical and quantum cases. It is therefore more difficult to give simple physical arguments to justify the dependences obtained, and we shall not attempt to do so here.

The above discussion was confined to the lowest Born approximation with inclusion of decay in intermediate states. In order for Eq. (7.30) to determine the true conditions for hydrodynamic behavior, we must verify that the contributions to $\Sigma_{\mu\nu}$ and $\Lambda_{\mu\nu}$ from higher-order diagrams are small throughout the range $\omega \ll \omega_{tr}$. In Sec. V, we argued that these higher contributions would be small asymptotically, as $k \to 0$, but we were unable to determine the precise expansion parameter. If we confine ourselves to the term of order $\epsilon_{\vec{k}}^2$, then the exact decay rate has the same form as in lowest Born approximation, except that the bare vertices $\Phi^{(i)}$ are replaced by dressed vertices $R^{(i)}$. In that case, the physical arguments given above are applicable, and Eq. (7.30) should describe the domain of validity of the hydrodynamic form. As mentioned in Sec. V, however, there are terms in the decay rate $\Gamma_{\vec{k}}$ with higher powers of $\epsilon_{\vec{k}}$, corresponding to an expansion in $\epsilon_{\vec{k}}/\tau^{n_0}$, which could invalidate the hydrodynamic result at frequencies which are lower than ω_{tr} given in Eq. (7.30). Since we are unable to make any precise statements about this expansion, we must leave the determination of the exact criterion for hydrodynamics as an open question. The difference between phonon and magnon systems remains in any case, since the situation described in Eq. (7.30), for which $\omega_{tr} \gg \Gamma_{th}$, is *possible* for magnons, but incorrect for phonons.

VIII. DAMPING OF UNIFORM MODE VIA ANISOTROPY

In this section we shall calculate the damping of the $k=0$ mode due to the presence of anisotropy. We consider only the case of uniaxial single-ion anisotropy. As one might expect, these calculations are quite similar to those for the isotropic model. In Sec. VIII A, we discuss the problem of kinematic consistency, requiring that for spin $\frac{1}{2}$ there should be no dynamical effects due to anisotropy. In Sec. VIII B, we discuss the various possible shapes of the scattering surface. This discussion leads directly to the classification of different regimes. In Secs. VIII C–VIII E, we carry out the calculations of the damping of the uniform mode in the first Born approximation in the various regimes described in Sec. VIII B. Finally, in Sec. VIII F, we discuss the self-consistency of the results of the lowest Born approximation.

A. Kinematic Consistency to Lowest Order in $1/z$

In order to carry out the calculation of the damping of the uniform mode due to anisotropy, we again use the lowest Born approximation as given in Eq. (2.38). Here, of course, we include the contributions of the single-ion anisotropy energy, $-D\sum_{\vec{r}}(S_{\vec{r}}^{z})^2$, both to the unperturbed free spin-wave Hamiltonian and to the spin-wave interaction terms. For simplicity, we confine our attention to the case of small anisotropy, and thus we keep only terms of lowest order in H_A/H_E, where H_A is the anisotropy field, $H_A = D(2S-1)$. As can be seen from Eq. (2.15), the effect of anisotropy on the noninteracting spin-wave spectrum is simply to change the dispersion relation to

$$\epsilon_{\vec{p}}^2 = \epsilon_0^2 + (\epsilon_{\vec{p}}^0)^2 , \tag{8.1}$$

where

$$H_E^2\, \epsilon_0^2 = 2H_A H_E + H_A^2 \approx 2H_A H_E \tag{8.2}$$

and

$$\epsilon_{\vec{p}}^0 = (1-\gamma_{\vec{p}}^2)^{1/2} . \tag{8.3}$$

Let us study interactions between spin waves in the anisotropic system. For simplicity, we shall consider only the interactions $\Phi^{(1)}$ and $\Phi^{(4)}$, since their behavior is typical. In the presence of anisotropy and at long wavelengths we have, for instance,

$$\Phi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim 4q_A + (\tfrac{1}{2}\vec{k}_3\cdot\vec{k}_4 - 2\epsilon_{\vec{3}}\epsilon_{\vec{4}}) , \tag{8.4a}$$

$$\Phi^{(4)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim 4q_A + (\tfrac{1}{2}\vec{k}_3\cdot\vec{k}_4 + 2\epsilon_{\vec{3}}\epsilon_{\vec{4}}) , \tag{8.4b}$$

as given by Eq. (A20), where

$$q_A = 2DS/H_E . \tag{8.5}$$

Note that the use of these coefficients will lead to a kinematic inconsistency for the special case of spin $\frac{1}{2}$. In this case, the anisotropy term $-D\sum_{\vec{r}}(S_{\vec{r}}^{z})^2$ reduces to a constant, and accordingly should not cause any spin-wave scattering. As has been shown elsewhere,[9,62] this kinematic property can be recovered by resumming over ladders made up of anisotropy vertices. Since summing over these ladders must be equivalent to an exact solution of a local potential problem, it is not surprising that we should thereby satisfy the spin kinematics. The most convenient way to perform this resummation is via a vertex renormalization. The procedure is almost identical to that in Ref. 62, except that here the energy dependence of the renormalization factor can be ignored.

The renormalization can be obtained simply as follows. We note that internal summations involving factors such as

$$N^{-1}\sum_{\vec{p}} m_{\vec{p}}^2 = N^{-1}\sum_{\vec{p}} l_{\vec{p}}^2 x_{\vec{p}}^2 , \tag{8.6a}$$

$$N^{-1}\sum_{\vec{p}} l_{\vec{p}}\, m_{\vec{p}}\, \gamma_{\vec{p}} , \tag{8.6b}$$

or

$$N^{-1}\sum_{\vec{p}} \gamma_{\vec{p}}^2 , \tag{8.6c}$$

introduce factors of $1/z$, and will be neglected. The only way we can obtain diagrams which have no additional internal summations giving factors of $1/z$, is by introducing repeated anisotropy scatterings. We then make the short-wavelength expansions

$$l_{\vec{p}}^2 = 1 + m_{\vec{p}}^2 \approx 1 , \tag{8.7a}$$

$$\gamma_{\vec{p}} \approx x_{\vec{p}} \approx 0 , \tag{8.7b}$$

$$\epsilon_{\vec{p}} \approx 1 , \tag{8.7c}$$

for all momenta which are not restricted to be small by occupation numbers. This procedure is most efficiently carried out in the a, b representation, and leads to the renormalized perturbation[62]

$$V = -N^{-1}\sum_{\vec{1}\vec{2}\vec{3}\vec{4}} \delta_{\vec{K}}(\vec{1}+\vec{2}-\vec{3}-\vec{4}) \Big(\frac{H_E}{2S}\, 2\gamma_{\vec{2}-\vec{4}}\, a_{\vec{1}}^\dagger b_{-\vec{2}} a_{\vec{3}} b_{-\vec{4}}^\dagger + \rho_A \frac{H_E}{2S} (\gamma_{\vec{2}}\, a_{\vec{1}}^\dagger b_{-\vec{2}} a_{\vec{3}} a_{\vec{4}} + \gamma_{\vec{3}+\vec{4}-\vec{2}}\, a_{\vec{1}}^\dagger b_{-\vec{2}} b_{-\vec{3}}^\dagger b_{-\vec{4}}^\dagger) + \rho_A D(a_{\vec{1}}^\dagger a_{\vec{2}}^\dagger a_{\vec{3}} a_{\vec{4}} + b_{\vec{1}}^\dagger b_{\vec{2}}^\dagger b_{\vec{3}} b_{\vec{4}}) \Big) , \tag{8.8}$$

where the renormalization factor ρ_A is

$$\rho_A = [1 - D/(H_A + H_E)]^{-1} . \tag{8.9}$$

The terms involving $\gamma_{\vec{1}-\vec{4}}$ or $\gamma_{\vec{2}-\vec{4}}$ are not renormalized in lowest order in $1/z$, because there are γ_i factors on both sides of the vertex. This prevents insertion of anisotropy ladders without concommitant factors of $1/z$. The renormalizations are shown in Figs. 6 and 7, from which the physics of the method should be apparent.

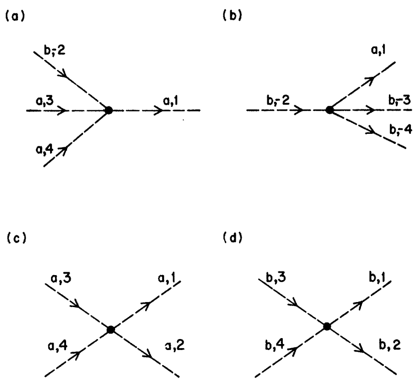

FIG. 6. Spin-wave interactions for the anisotropic system in the a-b representation. Scatterings of types (a) and (b) are due exclusively to exchange interactions, whereas those of types (c) and (d) are due solely to anisotropy.

With these renormalized potentials we find that in the zero-wave-vector limit $\Phi^{(1)}$ and $\Phi^{(4)}$ become

$$\Phi^{(1)} \sim \left(\frac{2H_A}{H_E}\right)\left(\frac{4D}{H_A+H_E-D}-\frac{H_A}{H_E}\right), \tag{8.10a}$$

$$\Phi^{(4)} \sim \left(\frac{2H_A}{H_E}\right)\left(\frac{4(H_A+H_E)}{H_A+H_E-D}+\frac{H_A}{H_E}\right). \tag{8.10b}$$

Note that these renormalized interactions are proportional to $H_A = D(2S-1)$ and hence they vanish for spin $\frac{1}{2}$, as is required by spin kinematics. For small anisotropy, the renormalized interactions are

$$\Phi^{(1)} \sim 0 + \mathcal{O}(H_A/H_E)^2 , \tag{8.11a}$$

$$\Phi^{(4)} \sim 8H_A/H_E + \mathcal{O}(H_A/H_E)^2 . \tag{8.11b}$$

In contrast, without the vertex renormalization, the interactions in the zero-wave-vector limit are found from Eq. (8.4) to be

$$\Phi^{(1)} \sim 4q_A - 4H_A/H_E , \tag{8.12a}$$

$$\Phi^{(4)} \sim 4q_A + 4H_A/H_E , \tag{8.12b}$$

where we have used Eq. (8.2). Here $\Phi^{(1)}$ and $\Phi^{(4)}$ do not vanish for spin $\frac{1}{2}$, because q_A is proportional to $2DS$ rather than, say to $2DS[1-(1/2S)] = H_A$. In fact, it is apparent from Eqs. (8.11) and (8.12) that to lowest order in H_A/H_E the vertex renormalization is equivalent to redefining q_A as

$$q_A = H_A/H_E . \tag{8.13}$$

Similar results are obtained for the other Φ coefficients. Henceforth, for simplicity we shall consider only the case of small anisotropy and shall rely on the above discussion to justify the use of Eq. (8.13) in place of Eq. (8.5).

B. Classification of Regimes

As we have seen in the isotropic Heisenberg model, various regimes may occur either because of the possibility of qualitatively different shapes of the scattering surface, or because of the need for a self-consistent treatment in the hydrodynamic regime. First, let us consider the possible shapes of the scattering surface, which for the uniform mode is given by

$$\epsilon_0 + \epsilon_{\vec{p}} = \epsilon_{\vec{p}-\vec{q}} + \epsilon_{\vec{q}} . \tag{8.14}$$

In Sec. III A, we have seen that for very small incident momentum the scattering surface consists of two disjoint pieces. By continuity, there must be a regime for the uniform mode in the anisotropic case which also has this property. To see this, we write Eq. (8.14) under the assumption that p is much larger than q in the form

$$\epsilon_0 - \epsilon_{\vec{q}} + \vec{q}\cdot\vec{\nabla}\epsilon_{\vec{p}} = 0 , \tag{8.15}$$

which has a solution of the form

$$q = 4\epsilon_0 \hat{q}\cdot\vec{v}_{\vec{p}}/[1-(\hat{q}\cdot\vec{v}_{\vec{p}})^2] . \tag{8.16}$$

In the expansion of the dispersion relation at long wavelengths, the correction terms to the linear spectrum can come either from the cubic term in p [as in Eq. (3.7)], or from the anisotropy:

$$\epsilon_{\vec{p}} = \tfrac{1}{2}p(1-gp^2+2\epsilon_0^2/p^2) . \tag{8.17}$$

For $p^3 \gg \epsilon_0$, the cubic correction term dominates the anisotropy correction term in both $\epsilon_{\vec{p}}$ and $v_{\vec{p}}$. Using (8.16), we then see that the maximum value of q (i.e., for $\hat{p}\cdot\hat{q}=1$) satisfies the ansatz for disjoint scattering surfaces $q=\epsilon_0/p^2 \ll p$. Since

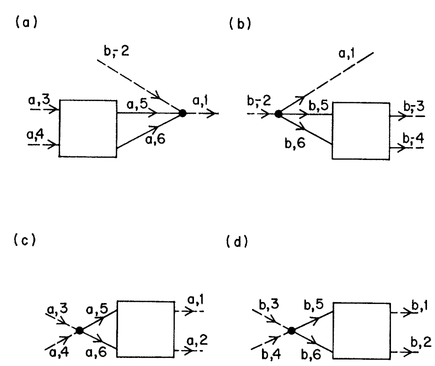

FIG. 7. Renormalized interactions for the anisotropic system in the a-b representation. Here the boxes represent ladders made up of anisotropy vertices only. Ladders of exchange scatterings involve higher powers of $1/z$ due to internal sums over factors of $\gamma_{\vec{k}}$.

p is a thermal momentum, this regime is characterized by $\tau^3 \gg \epsilon_0$. Now let us see if there is a regime involving disjoint surfaces in which the anisotropy correction dominates. From Eq. (8.17) we see that this requires that $p^2 \ll \epsilon_0$, in which case $v_{\vec{p}} \approx 1 - 2\epsilon_0^2/p^2$. But according to Eq. (8.16), the values of q then range between ϵ_0 and p^2/ϵ_0 depending upon the orientation of $\vec{v}_{\vec{p}}$. Since the geometric mean of these two values is of order p, it is clear that our ansatz of disjoint surfaces is violated. The second regime occurs when the scattering surfaces coalesce. But since we obtain different forms depending on how ϵ_0 compares with τ, it is clear that the second regime occurs when $\epsilon_0 \ll \tau \ll \epsilon_0^{1/3}$. Finally, there is the regime when $\tau \ll \epsilon_0$ and the scattering surface consists of a single component. In this low-temperature regime the dispersion relation is dominated by the anisotropy and hence is quadratic in the momentum. As in the case of a ferromagnet, one does not obtain disjoint scattering surfaces for this type of dispersion relation. Thus we expect the following three regimes:

$$\text{A}': \quad \epsilon_0 \ll \tau^3 , \tag{8.18a}$$

$$\text{B}': \quad \tau^3 \ll \epsilon_0 \ll \tau , \tag{8.18b}$$

$$\text{C}': \quad \tau \ll \epsilon_0 . \tag{8.18c}$$

Already, at this stage of the calculation, we see an analogy with the isotropic case, since these regimes are identical to those of Eq. (3.14) when the dispersion relation for the anisotropic case is replaced by the isotropic dispersion relation.

In all three regimes we shall calculate the decay rate of an on-resonance zero-wave-vector magnon using the formula of Eq. (2.38), which becomes in this case

$$\begin{aligned}\Gamma_0 \equiv \hbar^{-1}\Sigma_{\alpha\alpha}(\vec{k}=0,\ \tilde{\omega}=\epsilon_0) &= \tfrac{1}{8}\pi\,\omega_E S^{-2}(1-e^{-\beta H_E \epsilon_0})\\ &\times(2\pi)^{-6}\int d\vec{p}\int d\vec{q}\, n_{\vec{p}}(1+n_{\vec{q}})(1+n_{\vec{r}})\\ &\times\delta(2\epsilon_0+2\epsilon_{\vec{p}}-2\epsilon_{\vec{r}}-2\epsilon_{\vec{q}})\\ &\times M_{22}(0,\ \vec{p},\ \vec{q},\ \vec{r}) ,\end{aligned} \tag{8.19}$$

where we have used $H_A \ll H_E$ to replace $H_A + H_E$ by H_E. From the asymptotic forms, Eqs. (A20) and (A23), we obtain the following expression for the matrix element, $M_{22}(0, \vec{p}, \vec{q}, \vec{r})$ at long wavelengths and on the energy shell:

$$M_{22}(0,\ \vec{p},\ \vec{q},\ \vec{r}) = 2\epsilon_0(\epsilon_{\vec{p}}\epsilon_{\vec{q}}\epsilon_{\vec{r}})^{-1}(5\epsilon_0^2+\epsilon_{\vec{p}}^2+\epsilon_{\vec{q}}^2+\epsilon_{\vec{r}}^2) . \tag{8.20}$$

A self-consistent calculation of the decay rate of off-resonance zero-wave-vector magnons will be given in Sec. VIII F.

C. Regime A′

The calculation of the decay rate of on-shell zero-wave-vector magnons in regime A′ is based on Eq. (8.19). In this regime the scattering surface consists of two equivalent components. Hence we express Γ_0 as twice the contribution from the term for which $q \approx 0$. Then we have $q \ll p$, so that to lowest order in q/p:

$$M_{22}(0,\ \vec{p},\ \vec{q},\ \vec{r}) = 4(\epsilon_0/\epsilon_{\vec{q}}) , \tag{8.21}$$

also both ϵ_0 and $\epsilon_{\vec{q}}$ are much less than τ, so that

$$(1-e^{-\beta H_E \epsilon_0})(1+n_{\vec{q}}) = \epsilon_0/\epsilon_{\vec{q}} . \tag{8.22}$$

Thus, using Eq. (8.15) to describe the scattering surface, we have

$$\begin{aligned}\Gamma_0 = (\omega_A/S^2)(2\pi)^{-5}\int d\vec{p}\int d\vec{q}\, n_{\vec{p}}(1+n_{\vec{p}})\\ \times\epsilon_{\vec{q}}^{-2}\delta(2\epsilon_0-2\epsilon_{\vec{q}}+\vec{q}\cdot\vec{v}_{\vec{p}}) .\end{aligned} \tag{8.23}$$

We shall take q_z along the direction of $\vec{v}_{\vec{p}}$, so that $\vec{q}\cdot\vec{v}_{\vec{p}} = qv_{\vec{p}}\cos\theta = qv_{\vec{p}}\nu$. The angular integrals are trivial, the result being

$$\Gamma_0 = 2\omega_A S^{-2}(2\pi)^{-3}\int_0^\infty p^2\,dp\, v_p^{-1}\int_0^{q_{\max}} q\,dq\, n_p(1+n_p)\epsilon_q^{-2} . \tag{8.24}$$

Here n_p is given by Eq. (3.42), $\epsilon_q \approx (\epsilon_0^2 + \frac{1}{4}q^2)^{1/2}$, and $v_p = 1 - 3gp^2$ [see Eq. (3.29)]. Also, $q_{\max}$, the largest possible value of q, is given by Eq. (8.16) as

$$q_{\max} = 2\epsilon_0/(1-v_p) \approx 2\epsilon_0/3gp^2 . \tag{8.25}$$

Thus we find

$$\Gamma_0 = -(\omega_A/S^2\pi^3)\int_0^\infty p^2 n_p(1+n_p)\ln(3gp^2)\,dp , \tag{8.26}$$

which leads to the result

$$\Gamma_0 = 3\omega_A S^{-2}(2\pi)^{-3}\tau^3(a|\ln\tau| + a' - \tfrac{8}{9}\pi^2\ln 2) , \tag{8.27}$$

where the constants a and a' are given in Eq. (3.31). It should be noted that the result in Eq. (8.27) can be obtained from Eq. (3.32), apart from numerical factors of order unity, by simply replacing the isotropic spin-wave energy by that appropriate to the anisotropic case, since we may write $\frac{1}{2}\omega_E\epsilon_0^2$ in place of ω_A.

D. Regime B′

Next we consider regime B′. Over most of the phase space one has $p \sim \tau \gg \epsilon_0$ and, similarly, $q \gg \epsilon_0$. In this case, the scattering surface is approximately given by

$$\epsilon_0 + \tfrac{1}{2}p = \tfrac{1}{2}q + \tfrac{1}{2}|\vec{p}-\vec{q}| . \tag{8.28}$$

However, the integrals have a divergence at small q, so that it is necessary to use a form which represents this region correctly. Accordingly, we use the form

$$\epsilon_{\vec{p}}^2 = \epsilon_0^2 + \tfrac{1}{4}p^2 \equiv \epsilon_p^2 . \tag{8.29}$$

We write $\vec{r} = \vec{p} - \vec{q}$, so that

$$d\vec{p}\, d\vec{q} = 8\pi^2 pqr\, dp\, dq\, dr \,, \tag{8.30}$$

which allows us to write Eq. (8.19) as

$$\Gamma_0 = (\omega_E/4S^2)\, \epsilon_0^2\, \tau^{-1}\, (2\pi)^{-3} \int p\, dp \int q\, dq \int r\, dr$$

$$\times n_p (1+n_r)(1+n_q)\,(\epsilon_p^2 + \epsilon_r^2 + \epsilon_q^2)$$

$$\times (\epsilon_p\, \epsilon_q\, \epsilon_r)^{-1}\, \delta(\epsilon_0 + \epsilon_p - \epsilon_q - \epsilon_r) \,. \tag{8.31}$$

Here we have neglected ϵ_0 in comparison to p, q, and r. As in Eq. (3.36c), we have

$$(1+n_r)(1+n_q) = (1+n_p)(1+n_r+n_q) \,. \tag{8.32}$$

But, because of the symmetry between $\vec{r}$ and $\vec{q}$, we can replace $1+n_r+n_q$ by the factor $1+2n_q$. Then using Eq. (8.29), we obtain

$$\Gamma_0 = (8\omega_A/\tau S^2\pi^3) \int_{\epsilon_0}^{\infty} d\epsilon_p\, n_p\,(1+n_p) \int_{\epsilon_0}^{\epsilon_p} d\epsilon_q (1+2n_q)$$

$$\times (\epsilon_p^2 + \epsilon_q^2 - \epsilon_p\epsilon_q) \,. \tag{8.33}$$

It is apparent, however, that this integral is the same as the first one in Eq. (3.47). Thus we have

$$\Gamma_0 = (3\omega_A/32\tau S^2\pi^3)\, [I_B(\epsilon_k = \epsilon_0) - \tfrac{64}{9}\, \tau^4\, \zeta(2)] \,, \tag{8.34}$$

which leads to the result

$$\Gamma_0 = (\omega_A \tau^3/2\pi^3 S^2)\, [b \ln(\tau/2\epsilon_0) + b' - \tfrac{2}{9}\pi^2] \,. \tag{8.35}$$

Again, apart from factors of order unity, one gets the correct result for the anisotropic case by merely inserting the anisotropic spin-wave energy into the results for the isotropic case.

E. Regime C′

Finally we consider regime C′. Here all the momenta are in the quadratic regime, i.e., for all momenta we have

$$\epsilon_{\vec{p}} \approx \epsilon_0\, (1 + p^2/8\epsilon_0^2) \,, \tag{8.36a}$$

$$n_{\vec{p}} \approx e^{-T_{AE}/T}\, e^{-\gamma' p^2} \,, \tag{8.36b}$$

where $k_B T_{AE} = H_E \epsilon_0$ and

$$\gamma' = H_E/8\epsilon_0 k_B T = (4\epsilon_0 \tau)^{-1} \,. \tag{8.37}$$

We now evaluate Eq. (8.19) to lowest order in T/T_{AE} by replacing the energies where possible by their zero-momentum value. Then, keeping only the minimum number of Boltzmann factors, we find

$$\Gamma_0 = (\omega_E/2S^2)(2\pi)^{-5}\, e^{-T_{AE}/T} \int d\vec{p} \int d\vec{q}\, e^{-\gamma' p^2}$$

$$\times \delta\,([p^2 - q^2 - (p-q)^2]/8\epsilon_0) \,. \tag{8.38}$$

The evaluation of this expression is elementary, the result being

$$\Gamma_0 = \left(\frac{16\omega_E \epsilon_0}{S^2\pi^3}\right)\left(\frac{k_B T}{H_E}\right)^2 \left(\frac{H_A}{H_E}\right) e^{-T_{AE}/T} \,. \tag{8.39}$$

This result does not bear quite as close an analogy with the isotropic case as the previous ones. One can reason that the four powers of momentum which give rise to the factor τ^4 in Eq. (3.64) now lead to the factor $(k_B T/H_E)^2 (H_A/H_E)$. After that, replacement of $\omega_E \epsilon_{\vec{k}}$ by $\omega_E \epsilon_0$ leads to the result, Eq. (8.39), for the anisotropic case apart from factors of order unity.

F. Self-Consistent Magnons

Now let us discuss the self-consistency problem. We shall study the various regimes starting at the highest temperature and working towards zero temperature. At sufficiently high temperatures, it is clear that our calculations are not self-consistent. For instance, if $\epsilon_0 \ll \tau^5$, then perforce $\epsilon_0 \ll \tau$, so that the energy of a thermal spin wave is predominantly due to exchange energy. Accordingly, it is reasonable to assume that its damping is due to exchange interactions, so that, using the results of Sec. III, we estimate the relaxation rate of a thermal spin wave to be of order $\Gamma_{\rm th} \sim \omega_E \tau^5$. Thus when $\epsilon_0 \ll \tau^5$, we have $\omega/\Gamma_{\rm th} \ll 1$, and the thermal energy widths are much larger than the energy of the uniform mode; this necessitates a self-consistent treatment analogous to that given in Sec. IV for the isotropic system. As the temperature is reduced, it is reasonable to assume that although anisotropy eventually plays a role, the energy width of thermal spin waves decreases. If this is so, then we can conclude that our calculations are self-consistent for $\tau^5 \ll \epsilon_0$.

Thus self-consistency need only be considered for $\epsilon_0 \ll \tau^5$. We shall calculate the decay rate as a function of frequency, assuming $\tilde{\omega} = \rho\epsilon_0$, with ρ of order unity. In this regime, the energy width of thermal spin waves dominates $\hbar\omega_E \epsilon_0$. As in the isotropic case, this circumstance enables the three-magnon creation and annihilation processes to be operative. The method for taking account of the damping of the intermediate states was discussed in detail for the isotropic case in Sec. IV. By an entirely analogous calculation, we find here

$$\Gamma_0(\omega) \equiv \Gamma(\vec{k}=0,\omega) = \tfrac{1}{8}\pi S^{-2}\omega_E \beta\hbar\omega (2\pi)^{-6} \int d\vec{p}\, n_{\vec{p}}$$

$$\times \int d\vec{q} (1+n_{\vec{q}})(1+n_{\vec{r}}) \int_{-\infty}^{+\infty} \varphi(\alpha)\, d\alpha$$

$$\times [\delta(2\tilde{\omega} - 2\alpha + 2\epsilon_{\vec{p}} - 2\epsilon_{\vec{r}} - 2\epsilon_{\vec{q}})\, M_{22}(0, \vec{p}, \vec{q}, \vec{r})$$

$$+ \delta(-2\tilde{\omega} - 2\alpha + 2\epsilon_{\vec{p}} - 2\epsilon_{\vec{r}} - 2\epsilon_{\vec{q}})\, M_{31}(0, \vec{p}, \vec{q}, \vec{r})] \,. \tag{8.40}$$

Substituting the asymptotic forms given by Eqs. (A20) and (A23) into the expressions for $M_{22}(0, \vec{p}, \vec{q}, \vec{r})$ and $M_{31}(0, \vec{p}, \vec{q}, \vec{r})$ given in Eqs. (2.39) and (4.4), respectively, we find

$$M_{22}(0, \vec{p}, \vec{q}, \vec{r}) = (\epsilon_{\vec{p}}\epsilon_{\vec{q}}\epsilon_{\vec{r}})^{-1}\{4\epsilon_0\epsilon_{\vec{p}}^2 - \Delta\epsilon[\epsilon_{\vec{p}}^2 + \epsilon_{\vec{q}}^2 + 2\epsilon_{\vec{p}}\epsilon_{\vec{q}} + 2\epsilon_{\vec{q}}(-\Delta\epsilon + \epsilon_{\vec{p}} - 2\epsilon_{\vec{q}}) + (-\Delta\epsilon + \epsilon_{\vec{p}} - 2\epsilon_{\vec{q}})^2]\}, \tag{8.41a}$$

$$M_{31}(0, \vec{p}, \vec{q}, \vec{r}) = (\epsilon_{\vec{p}}\epsilon_{\vec{q}}\epsilon_{\vec{r}})^{-1}\{4\epsilon_0\epsilon_{\vec{p}}^2 + \delta\epsilon[\epsilon_{\vec{p}}^2 + \epsilon_{\vec{q}}^2 + 2\epsilon_{\vec{p}}\epsilon_{\vec{q}} + 2\epsilon_{\vec{q}}(-\delta\epsilon + \epsilon_{\vec{p}} - 2\epsilon_{\vec{q}}) + (-\delta\epsilon + \epsilon_{\vec{p}} - 2\epsilon_{\vec{q}})^2]\}, \tag{8.41b}$$

where $\Delta\epsilon = \epsilon_0 + \epsilon_{\vec{p}} - \epsilon_{\vec{r}} - \epsilon_{\vec{q}}$ and $\delta\epsilon = \epsilon_{\vec{p}} - \epsilon_0 - \epsilon_{\vec{r}} - \epsilon_{\vec{q}}$. The terms independent of $\Delta\epsilon$ and $\delta\epsilon$ contribute on resonance and give rise to the results found in Sec. VIII C. The other terms, since they are of relative order α/ϵ_0, potentially dominate, but, as in the isotropic case, there is a partial cancellation between the matrix elements M_{22} and M_{31}, and the decay rate remains of order ϵ_0^2. Accordingly, it is correct to keep only the dominant terms in $\Delta\epsilon$ and $\delta\epsilon$:

$$M_{22} \sim 4(\epsilon_0 - \Delta\epsilon)/\epsilon_{\vec{q}}, \tag{8.42a}$$

$$M_{31} \sim 4(\epsilon_0 - \delta\epsilon)/\epsilon_{\vec{q}}. \tag{8.42b}$$

Expressing $\Gamma_0(\omega)$ as twice the contribution from the component of the scattering surface near $q = 0$, and using the asymptotic forms of Eq. (8.42) we find

$$\Gamma_0(\omega) = \int_{-\infty}^{+\infty} \varphi(\alpha)\, d\alpha\, [\epsilon_0^{-1}(\tilde{\omega} - \alpha)\,\Delta\omega^0(\tilde{\omega} - \alpha) + \epsilon_0^{-1}(\tilde{\omega} + \alpha)\,\Delta\omega^0(-\tilde{\omega} - \alpha)], \tag{8.43}$$

where

$$\Delta\omega^0(\xi) = (\rho\omega_A/S^2)\,(2\pi)^{-5}\int d\vec{p}\, n_{\vec{p}}(1 + n_{\vec{p}}) \times \int_{q\approx 0} d\vec{q}\, \epsilon_{\vec{q}}^{-2}\, \delta(2\xi - \vec{q}\cdot v_{\vec{p}} - 2\epsilon_{\vec{q}}). \tag{8.44}$$

Since $\varphi(\alpha)$ is essentially an even function of α, we can write Eq. (8.43) as

$$\Gamma_0(\omega) = \int_{-\infty}^{+\infty} \Delta\omega^0(\alpha)\, d\alpha\, [\varphi(\alpha - \omega) - \varphi(\alpha + \omega)]\left(\frac{\alpha}{\epsilon_0}\right), \tag{8.45a}$$

$$\Gamma_0(\omega) = -\int_{-\infty}^{+\infty} \Delta\omega^0(\alpha)\, d\alpha \left(\frac{2\alpha\omega}{\epsilon_0}\right)\left(\frac{d\varphi(\alpha)}{d\alpha}\right). \tag{8.45b}$$

Let us now evaluate $\Delta\omega^0(\xi)$. In analogy with Eq. (8.24), we have

$$\Delta\omega^0(\xi) = 2\rho\omega_A S^{-2}(2\pi)^{-3}\int_0^\infty p^2\, dp\, v_p^{-1} n_p(1 + n_p) \times \int_{q_{\min}}^{q_{\max}} q\, dq\, \epsilon_q^{-2}, \tag{8.46}$$

where $q_{\min}$ and $q_{\max}$ are the extreme values of q for which the δ-function condition in Eq. (8.44) can be satisfied. This condition is

$$2\xi - 2\epsilon_{\vec{q}} = v_p\, \nu(4\epsilon_{\vec{q}}^2 - 4\epsilon_0^2)^{1/2}, \tag{8.47}$$

where ν is defined in Eq. (3.22a), and we have approximated the dispersion relation by Eq. (8.29).

Solving the quadratic equation for $\epsilon_{\vec{q}}$ implied by (8.47), one determines the maximum and minimum values of $\epsilon_{\vec{q}}$ for $\xi > 0$ as

$$\epsilon_{\max} \equiv \epsilon(q_{\max}) = \xi/(1 - v_p), \tag{8.48a}$$

$$\epsilon_{\min} \equiv \epsilon(q_{\min}) = \xi/(1 + v_p). \tag{8.48b}$$

For $\xi < 0$, the δ-function condition cannot be satisfied. In obtaining these results we have assumed $\xi \gg \epsilon_0$, since the variable α in Eq. (8.42) may be considered to be of order τ^5. Inserting these results into Eq. (8.45), we find that

$$\Delta\omega^0(\xi) = 3\omega_A S^{-2}(\tau/2\pi)^3 \rho(a|\ln\tau| + a'), \quad \xi > 0 \tag{8.49a}$$

$$\Delta\omega^0(\xi) = 0, \quad \xi < 0. \tag{8.49b}$$

We can now evaluate Eq. (8.45b) by integrating the second term by parts, followed by use of Eq. (D12):

$$\Gamma_0(\omega) = 3\omega_A S^{-2}(\tau/2\pi)^3 \rho^2(a|\ln\tau| + a'), \tag{8.50}$$

where a and a' are as in Eq. (3.31). Thus we see that the damping of intermediate states does produce a slight change relative to Eq. (8.27). The dominant term in $\ln\tau$ is invariant, however.

IX. SUMMARY AND CONCLUSION

A. Summary of Decay-Rate Calculations

The decay rate of antiferromagnetic magnons has been calculated in a number of low-temperature regimes for a bcc lattice, with interactions between nearest neighbors on opposite sublattices. Let us summarize the results of these calculations.

In the quantum region for the isotropic model there are four regimes, depending on the relation between the reduced spin-wave energy $\epsilon_{\vec{k}} = E_{\vec{k}}/2JzS$ and the dimensionless temperature $\tau = k_B T/JzS$:

Regime A: $\epsilon_{\vec{k}} \ll \tau^3 \ll 1$,

$$\Gamma_{\vec{k}} = \hbar^{-1}\Sigma''(\vec{k}, \omega_E\epsilon_{\vec{k}}) = (2\omega_E/S^2)\epsilon_{\vec{k}}^2\tau^3(2\pi)^{-3}(a|\ln\tau| + a'); \tag{9.1a}$$

Regime B: $\tau^3 \ll \epsilon_{\vec{k}} \ll \tau \ll 1$,

$$\Gamma_{\vec{k}} = (8\omega_E/3S^2)\epsilon_{\vec{k}}^2\tau^3(2\pi)^{-3}[b\ln(\tau/k) + b']; \tag{9.1b}$$

Regime C: $\tau \ll \epsilon_{\vec{k}} \ll \tau^{1/3} \ll 1$,

$$\Gamma_{\vec{k}} = (\pi\omega_E/108S^2)\epsilon_{\vec{k}}\tau^4; \tag{9.1c}$$

Regime D: $\tau^{1/3} \ll \epsilon_{\vec{k}} \ll 1$,

$$\Gamma_{\vec{k}} = (\omega_E/2S^2\pi^3)\,\tau^5\zeta(5)\,[g(\hat{k})\,\epsilon_{\vec{k}}^2]^{-1}. \tag{9.1d}$$

In these formulas $\omega_E = 2JzS/\hbar$, $g(\hat{k})$ is an angular function defined in Eq. (3.8), the numerical constants a, a', b, and b' are given in Eqs. (3.31) and (3.49), and $\zeta(5) = 1.037$. In all four regimes the calculations are stable with respect to self-consistent inclusion of damping in intermediate states. Regime A is the hydrodynamic region, in which the mass operator has also been calculated off-resonance, i.e., for $\hbar\omega \neq E_{\vec{k}}$, and the result is given in Eq. (4.24). The corrections to the above formulas are of relative order $(zS)^{-1}$, τ, and $\epsilon_{\vec{k}}$, and are therefore small at long wavelengths and low temperatures, as long as $zS \gg 1$.

In the quantum-mechanical model with uniaxial single-ion anisotropy there are four regimes, depending on the relation between τ and the dimensionless anisotropy energy ϵ_0 defined by

$$\omega_E^2\epsilon_0^2 = 2\omega_A\omega_E + \omega_A^2 \sim 2\omega_A\omega_E \ , \tag{9.2}$$

where $\hbar\omega_A$ is the anisotropy energy which we assume to be much less than the exchange energy $\hbar\omega_E$. The damping of the uniform $(k=0)$ mode is given by

Regime A_1': $\epsilon_0 \ll \tau^5 \ll 1$,

$$\Gamma_0 = (3\omega_A\tau^3/S^2)(2\pi)^{-3}(a|\ln\tau| + a') \ ; \tag{9.3a}$$

Regime A_2': $\tau^5 \ll \epsilon_0 \ll \tau^3 \ll 1$,

$$\Gamma_0 = (3\omega_A\tau^3/S^2)(2\pi)^{-3}[a|\ln\tau| + a' - (8\pi^2/9)\ln 2] \ ; \tag{9.3b}$$

Regime B': $\tau^3 \ll \epsilon_0 \ll \tau$,

$$\Gamma_0 = (\omega_A\tau^3/2\pi^3S^2)[b\ln(\tau/2\epsilon_0) + b' - \tfrac{2}{9}\pi^2] \ ; \tag{9.3c}$$

Regime C': $\tau \ll \epsilon_0 \ll 1$,

$$\Gamma_0 = 16\left(\frac{\omega_0}{S^2\pi^3}\right)\left(\frac{k_BT}{H_E}\right)^2\left(\frac{H_A}{H_E}\right)e^{-T_{AE}/T} \ ; \tag{9.3d}$$

where $\omega_0 = \hbar^{-1}H_E\epsilon_0 = \hbar^{-1}k_BT_{AE} \sim (2\omega_A\omega_E)^{1/2}$.

The classical low-temperature regime is obtained in the limit $\hbar \to 0$, $J \to 0$, $S \to \infty$ with $\hbar S$, $k_BT_0 = 2zJS^2$, and $\omega_E = 2zJS/\hbar$ remaining finite (T_0 is of order T_N). For the isotropic model, the calculations may be performed at long wavelengths, and the behavior is hydrodynamic for any $k \ll 1$. We find

$$\Gamma_{\vec{k}} = (4\eta/3\pi)\,\omega_E(T/T_0)^2\epsilon_{\vec{k}}^2, \quad T/T_0 \ll 1 \tag{9.4}$$

where the numerical constant η is defined in Eq. (6.15). Corrections to Eq. (9.4) are of relative order T/T_0 and $\epsilon_{\vec{k}}$. Here again the decay rate has been calculated off resonance, for $\omega/\omega_E\epsilon_{\vec{k}}$ finite, and is stable with respect to the inclusion of damping in intermediate states.

B. Comparison with Other Authors's Results

For the isotropic model, calculations of the damping were performed previously by Kaganov, Tsukernik, and Chupis,[16] who found

$$\Gamma_{\vec{k}} \sim S^{-2}\omega_E\tau^5 \ , \tag{9.5}$$

and by Tani,[17] who found

$$\Gamma_{\vec{k}} \sim S^{-2}\omega_E\tau^2\epsilon_{\vec{k}} \ . \tag{9.6}$$

These two calculations do not agree with our results for any regime [see Eq. (9.1)]. We believe that in both cases the error lies in the use of an incorrect magnon-magnon interaction at long wavelengths. Very recently, Solyom[18] has evaluated the decay rate using the formalism of Vaks *et al.*,[28] obtaining the result

$$\Gamma_{\vec{k}} \sim \omega_ES^{-2}\epsilon_{\vec{k}}^4\tau^2\ln(\tau/\epsilon_{\vec{k}}) \ , \quad \epsilon_{\vec{k}} \ll \tau \ll 1 \tag{9.7a}$$

$$\Gamma_{\vec{k}} \sim \omega_ES^{-2}\epsilon_{\vec{k}}^4\tau^2 \ , \qquad \tau \ll \epsilon_{\vec{k}} \ll 1 \ . \tag{9.7b}$$

Although the formula for the decay rate used by Solyom [his Eq. (3.10)] is very similar to our lowest Born approximation, there seem to be some algebraic errors in his work which make a detailed comparison difficult. The recent calculation of Cottam and Stinchcombe[100] predicts a decay rate proportional to T^2 at low temperatures, which does not agree with the results in any of our regimes. Since the coefficient is not evaluated by these authors, it is difficult to pinpoint the source of the discrepancy.

For the anisotropic model, there have been several calculations of the damping of the uniform $(k=0)$ mode. For instance, Urushadze[43] found

$$\Gamma_0 \approx (\omega_A^2/\omega_E)\tau, \quad T \gg T_{AE} \ , \tag{9.8}$$

whereas Genkin and Fain[44] obtain the result (using our definition of ω_E)

$$\Gamma_0 = 32\omega_A^2\tau\omega_E^{-1}e^{-T_{AE}/T}, \quad T \ll T_{AE} \ . \tag{9.9}$$

Both of these results are based on a Golden Rule calculation of the scattering rate in a Boltzmann equation, and hence these were exclusively on-resonance calculations. Kashcheev[45] was the first to calculate the mass operator of a Green's function, thus retaining the possibility of discussing off-resonance processes. However his result, which is applicable to nonzero values of k, has a very unphysical behavior in the limit of small anisotropy:

$$\Gamma_{\vec{k}} \sim \tau^8\omega_E\epsilon_{\vec{k}}(\omega_E/\omega_A)^2 \ . \tag{9.10}$$

By using an equation-of-motion method, Tani[17] has been able to allow renormalized, but undamped, propagation in intermediate states. By a similar method, Kawasaki[46] has essentially reproduced Tani's results, and has evaluated the numerical constants, finding

$$\Gamma_0 = (27z^2/2^{11}\pi^3S^2)\,\omega_A\tau e^{-T_{AE}/T}, \quad T \ll T_{AE} \tag{9.11a}$$

$$\Gamma_0 = (27z^2/2^{25/2}\pi^3S^2)\,\omega_0\tau^2 \ , \qquad T_{AE} \ll T \ . \tag{9.11b}$$

One of the present authors[47] previously investigated the damping in the lowest Born approximation using the Dyson-Maleev formalism, obtaining the results

$$\Gamma_0 = (8\omega_A\omega_0/\pi^3S^2\omega_E)\,\tau^2e^{-T_{AE}/T}, \quad T \ll T_{AE} \tag{9.12a}$$

$$\Gamma_0 = [5\omega_A\zeta(3)/2\pi^2S^3]\tau^3 \ , \qquad T_{AE} \ll T \ . \tag{9.12b}$$

Finally, Solyom[18] has also calculated the damping; his result is unphysical, because the damping does not vanish for $k=0$ in the limit of vanishing anisotropy:

$$\Gamma_{\bar{k}} \sim \epsilon_{\bar{k}} S^{-2}\tau^2\delta^{-1/2}, \quad \epsilon_{\bar{k}} \ll \tau \ll 1 \tag{9.13}$$

where δ is the double-ion anisotropy constant.

It is clear that the previous authors have not treated the shape of the scattering surface correctly, nor have they considered the self-consistency problem adequately. In this connection we note that while it is important to take account of the damping of intermediate states, the (real) renormalization of the magnon energies is rather unimportant at low temperatures. The various calculations seem to differ because of differences in the matrix elements (analogous to our M_{22}) which are used. This comment does not appear to apply to Ref. 47, where the result Eq. (9.12a) agrees with the present one, apart from numerical factors. The other result, Eq. (9.12b), is slightly different, the logarithmic cutoff factors having been lost by an incorrect treatment of the scattering surface. It is probable that the incorrect results of other authors are again due to an insufficiently delicate handling of the cancellations involved in computing the matrix elements. Since these cancellations are crucial for obtaining physically correct results, we have discussed them in great detail in the present paper. As remarked above, it seems likely to us that a different formalism could be found in which the long-wavelength form of the magnon interaction can be obtained in a simpler way, without the necessity of these delicate cancellations.

C. Experimental Prospects

Two of the present authors have discussed elsewhere[13] the prospects for observing the hydrodynamic damping of spin waves experimentally, using the technique of neutron scattering. Since this damping is small at low temperatures, it is necessary to perform the experiments near the Néel point, where the present results do not apply. Similarly, the damping in the nonhydrodynamic low-temperature regimes is probably too small to be observed by neutron scattering. One method which seems more hopeful is the parallel-pumping technique,[49,50] for which the resolution is much greater. A difficulty, however, is that this method is confined to very long wavelengths, where microscopic inhomogeneities, imperfections, and dipolar interactions[101] may play a large role. In any event, before detailed predictions can be made, the present work must be extended to treat finite momentum and anisotropy together, which is probably not difficult in principle, using the methods employed here. In addition, it might be necessary to use a more realistic model including effects such as anisotropic exchange, dipolar interactions, and higher-order anisotropy terms.

For the uniform mode in the anisotropic model, the situation is more hopeful, because it is possible that under the right conditions one might observe the intrinsic width of the antiferromagnetic resonance mode at the frequency $\omega_0 \sim (2\omega_A\omega_E)^{1/2}$. In fact, we can compare the order of magnitude of our results with the experiments of Johnson and Nethercot[102] for MnF_2, although there the anisotropy is due mainly to dipole-dipole interactions, rather than to a single-ion crystalline field.[103,104] Nonetheless, for the crude comparison between theory and experiment which we make here, perhaps the microscopic details of the anisotropy are not so important, and we shall use our results in terms of the observed anisotropy field. From the data of Johnson and Nethercot[102] we see that the linewidth does not vanish in the zero-temperature limit. This behavior would imply that other mechanisms, such as strains and impurities unfortunately are dominant in the low-temperature regime, where our theory might be expected to be the most reliable. Consequently, the only reasonable procedure is to compare our values of $2\Gamma_0$ with the experimental values of $\Delta\nu(T)-\Delta\nu(0)$ (note that their $\Delta\nu$ is the full width at half-intensity), so that we eliminate the breadth due to the zero-temperature mechanisms. It is clear that $\Delta\nu(T)-\Delta\nu(0)$ is too small to be defined precisely by the data. Nevertheless, from Fig. 9 of Ref. 102 we estimate that at $T/T_N=0.5$, $(h/g\mu_B)[\Delta\nu(T)-\Delta\nu(0)]=200$ G, where g is the appropriate g value of Mn spins in MnF_2, and μ_B is the Bohr magneton. We use our formula, Eq. (8.35), taking $H_A/g\mu_B=8800$ G, $\tau=2k_BT/H_E=0.95$, and $\epsilon_0\sim(2H_A/H_E)^{1/2}=0.17$, which then gives $2\Gamma_0=180$ G. The excellent agreement with the experimental value is clearly fortuitous, but it is gratifying that the orders of magnitude are in agreement.

Recently, Seehra and Castner[105] have studied the antiferromagnetic resonance linewidth in copper formate tetrahydrate. They find that the linewidth varies as $T^{3.3}$, in qualitative agreement with our results [see Eq. (9.3)]. Since they attribute this linewidth to a large Dzyaloshinskii-Moriya interaction, it is clear that again our model does not really apply. As mentioned above, however, it is possible that the decay rate depends only on the value of the anisotropy field and not on the details of the underlying microscopic mechanism. To the extent that such an argument is tenable, the experimental data confirm our calculation.

Further experimental studies of linewidths either via antiferromagnetic resonance (for anisotropic systems), parallel pumping, or inelastic neutron scattering would be desirable to check our calculations. Possibly, a fruitful line of investigation would be to study two-dimensional systems,[51] where the decay rates may be larger than those found for three-dimensional systems.

D. Conclusion

We shall conclude by restating the principal results of the present work:

(i) The decay rate of magnons in antiferromagnets has been calculated self-consistently at long wavelengths and low temperatures for isotropic systems in classical and quantum low-temperature regimes. For the anisotropic model, the decay rate was calculated quantum mechanically at $k=0$. In all cases, the decay rate is small in comparison to the spin-wave energy, so that spin waves are appropriate elementary excitations.

(ii) In the isotropic model, the decay rate at long wavelengths has the form predicted by hydrodynamics, namely, $\Gamma_{\vec{k}} \propto \epsilon_{\vec{k}}^2$. This result is valid for $\epsilon_{\vec{k}} \ll (k_B T/2JzS)^3$ quantum mechanically, and for $\epsilon_{\vec{k}} \ll 1$ classically. For very small anisotropy in the quantum low-temperature limit, a similar result is found: $\Gamma_0 \propto \epsilon_0^2$.

(iii) The dynamic correlation functions for the transverse components of the staggered and total spin have been calculated at long wavelengths and low frequencies in the isotropic model at low temperatures. They are also found to have the form predicted by hydrodynamics. A comparison of the microscopic and macroscopic results yields expressions for the thermodynamic parameters and transport coefficients at low temperatures in terms of microscopic quantities in both the classical and quantum regimes.

Within the approximation of low density of spin deviations, i.e., for $\tau \ll 1$ and $(zS)^{-1} \ll 1$, the $(+-)$ spin Green's functions may be written in the usual form for single-particle boson Green's functions. The mass operator $\underline{\Sigma}^S$ is different from the boson mass operator $\underline{\Sigma}$, namely, $\underline{\Sigma}^S = \underline{\Sigma} + \underline{G}_0^{-1}\underline{\Lambda}$. In higher orders in $(zS)^{-1}$ this picture breaks down, and then the transverse spin-correlation functions are expressed in terms of vertex functions at long wavelengths and low temperatures, in a way reminiscent of the theory of Fermi liquids.[96,63]

(iv) The magnons obtained in the Dyson-Maleev formalism are found to interact weakly in the isotropic system, both on and off the energy shell, and hence the self-consistent inclusion of damping in intermediate states leads only to perturbative effects. The magnon interactions turn out to be analogous to those present in a ferromagnet. In particular, in the low-temperature limit the renormalized interactions are Hermitian on resonance and on the energy shell, at least within low-order perturbation theory. This property is shown to hold to all orders in $1/S$ for the two-spin-wave t matrix of the ferromagnet.

(v) Spin-wave damping and spin-correlation functions have also been calculated using the Holstein-Primakoff boson representation. As one might expect, the boson self-energies are not identical in the two formalisms. However, physically meaningful results, in particular, the decay rate on resonance and the spin-correlation functions, agree in the two formalisms, when the corrections due to the self-consistent inclusion of damping in intermediate states are estimated to be small. At the longest wavelengths, however, the Holstein-Primakoff expressions are not self-consistent, and the lowest Born approximation contains large corrections, of relative order $(k_B T/2JzS)^5/S^2\epsilon_{\vec{k}}$, quantum mechanically and of order $(T/T_0)^2/\epsilon_{\vec{k}}$ classically. When these quantities become large, the Holstein-Primakoff bare spin excitations do not provide a convenient basis for calculating the damping of the spin waves.

APPENDIX A: ASYMPTOTIC FORMS OF INTERACTION COEFFICIENTS

In this Appendix we shall give the asymptotic forms of the interaction coefficients in the limit when one or more of the interacting spin waves has very small momentum. These results will be of use for both the classical and quantum-mechanical calculations described in the main body of this paper. Unless explicitly stated to the contrary, our discussion will be confined to the isotropic model. In obtaining the results of this Appendix the following relations are useful:

$$\epsilon_{\vec{k}} = (1-\gamma_{\vec{k}}^2)^{1/2} \sim \tfrac{1}{2}k \,, \qquad k \to 0 \qquad \text{(A1a)}$$

$$\gamma_{\vec{k}} = (1-\epsilon_{\vec{k}}^2)^{1/2} \sim 1 - \tfrac{1}{2}\epsilon_{\vec{k}}^2 \,, \qquad k \to 0 \qquad \text{(A1b)}$$

$$x_{\vec{k}} = [(1-\epsilon_{\vec{k}})/(1+\epsilon_{\vec{k}})]^{1/2} \sim 1 - \epsilon_{\vec{k}}, \qquad k \to 0 \qquad \text{(A1c)}$$

$$\gamma_{\vec{k}} x_{\vec{k}} = 1 - \epsilon_{\vec{k}} \,, \qquad \text{(A1d)}$$

$$\gamma_{\vec{k}} / x_{\vec{k}} = 1 + \epsilon_{\vec{k}} \,, \qquad \text{(A1e)}$$

$$\gamma_{\vec{k}+\vec{q}} \sim \gamma_{\vec{k}} - \tfrac{1}{2}\gamma_{\vec{k}}^{-1}\epsilon_{\vec{k}}\vec{v}_{\vec{k}} \cdot \vec{q} \,, \qquad q \to 0 \qquad \text{(A1f)}$$

$$x_{\vec{k}+\vec{q}} \sim x_{\vec{k}} - \tfrac{1}{2}\gamma_{\vec{k}}^{-1}(1+\epsilon_{\vec{k}})^{-1}\vec{v}_{\vec{k}} \cdot \vec{q} \,, \qquad q \to 0 \qquad \text{(A1g)}$$

where $\vec{v}_{\vec{k}}$ is defined as $\vec{v}_{\vec{k}} \equiv 2\nabla_{\vec{k}}\epsilon_{\vec{k}}$, so that $\vec{v}_{\vec{k}} \to \hat{k}$ as $k \to 0$.

First we give asymptotic forms of the interaction coefficients[70] when a single *incoming* momentum $\vec{k}_3$ approaches zero. (We shall frequently denote this asymptotic limit by the symbol Φ_{in}.) We find for Φ_{in}:

$$\begin{aligned}\Phi^{(3)}_{\vec{1},\vec{2},\vec{4},-\vec{3}} = \Phi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = \epsilon_{\vec{3}}\,(\gamma_{\vec{1}}x_{\vec{2}}x_{\vec{4}} + \gamma_{\vec{2}}x_{\vec{1}}x_{\vec{4}} - \gamma_{\vec{2}}x_{\vec{2}} - \gamma_{\vec{1}}x_{\vec{1}}) \\ + \tfrac{1}{2}\gamma_{\vec{2}}^{-1}\epsilon_{\vec{2}}\vec{v}_{\vec{2}} \cdot \vec{k}_3(x_{\vec{2}} + x_{\vec{1}}x_{\vec{4}}) \\ + \tfrac{1}{2}\gamma_{\vec{1}}^{-1}\epsilon_{\vec{1}}\vec{v}_{\vec{1}} \cdot \vec{k}_3(x_{\vec{1}} + x_{\vec{2}}x_{\vec{4}}) \,,\end{aligned} \qquad \text{(A2)}$$

$$\begin{aligned}\Phi^{(7)}_{\vec{1},\vec{2},-\vec{3},\vec{4}} = \Phi^{(3)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = \epsilon_{\vec{3}}(\gamma_{\vec{1}}x_{\vec{1}}x_{\vec{4}} + \gamma_{\vec{2}}x_{\vec{2}}x_{\vec{4}} - \gamma_{\vec{1}}x_{\vec{2}} - \gamma_{\vec{2}}x_{\vec{1}}) \\ - \tfrac{1}{2}\gamma_{\vec{1}}^{-1}\epsilon_{\vec{1}}\vec{v}_{\vec{1}} \cdot \vec{k}_3(x_{\vec{1}}x_{\vec{4}} + x_{\vec{2}}) \\ - \tfrac{1}{2}\gamma_{\vec{2}}^{-1}\epsilon_{\vec{2}}\vec{v}_{\vec{2}} \cdot \vec{k}_3(x_{\vec{2}}x_{\vec{4}} + x_{\vec{1}}) \,,\end{aligned} \qquad \text{(A3)}$$

$$\Phi^{(4)}_{\vec{1},\vec{2},\vec{4},-\vec{3}} = \Phi^{(2)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = \epsilon_{\vec{3}}(\gamma_{\vec{1}}x_{\vec{1}}x_{\vec{2}} + \gamma_{\vec{2}} - \gamma_{\vec{1}}x_{\vec{4}} - \gamma_{\vec{2}}x_{\vec{1}}x_{\vec{2}}x_{\vec{4}}) - \tfrac{1}{2}\gamma_{\vec{1}}^{-1}\epsilon_{\vec{1}}\vec{v}_{\vec{1}}\cdot\vec{k}_3(x_{\vec{1}}x_{\vec{2}} + x_{\vec{4}}) - \tfrac{1}{2}\gamma_{\vec{2}}^{-1}\epsilon_{\vec{2}}\vec{v}_{\vec{2}}\cdot\vec{k}_3(1 + x_{\vec{1}}x_{\vec{2}}x_{\vec{4}}) , \tag{A4}$$

$$\Phi^{(6)}_{\vec{1},\vec{2},-\vec{3},\vec{4}} = \Phi^{(4)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = \epsilon_{\vec{3}}(\gamma_{\vec{1}} + \gamma_{\vec{2}}x_{\vec{1}}x_{\vec{2}} - \gamma_{\vec{2}}x_{\vec{4}} - \gamma_{\vec{1}}x_{\vec{1}}x_{\vec{2}}x_{\vec{4}}) + \tfrac{1}{2}\gamma_{\vec{1}}^{-1}\epsilon_{\vec{1}}\vec{v}_{\vec{1}}\cdot\vec{k}_3(1 + x_{\vec{1}}x_{\vec{2}}x_{\vec{4}}) + \tfrac{1}{2}\gamma_{\vec{2}}^{-1}\epsilon_{\vec{2}}\vec{v}_{\vec{2}}\cdot\vec{k}_3(x_{\vec{1}}x_{\vec{2}} + x_{\vec{4}}) , \tag{A5}$$

$$\Phi^{(5)}_{\vec{1},\vec{2},\vec{4},-\vec{3}} = \Phi^{(8)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = \epsilon_{\vec{3}}(\gamma_{\vec{1}}x_{\vec{1}}x_{\vec{4}} + \gamma_{\vec{2}}x_{\vec{2}}x_{\vec{4}} - \gamma_{\vec{1}}x_{\vec{2}} - \gamma_{\vec{2}}x_{\vec{1}}) + \tfrac{1}{2}\gamma_{\vec{1}}^{-1}\epsilon_{\vec{1}}\vec{v}_{\vec{1}}\cdot\vec{k}_3(x_{\vec{2}} + x_{\vec{1}}x_{\vec{4}}) + \tfrac{1}{2}\gamma_{\vec{2}}^{-1}\epsilon_{\vec{2}}\vec{v}_{\vec{2}}\cdot\vec{k}_3(x_{\vec{1}} + x_{\vec{2}}x_{\vec{4}}) , \tag{A6}$$

$$\Phi^{(9)}_{\vec{1},\vec{2},-\vec{3},\vec{4}} = \Phi^{(5)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = \epsilon_{\vec{3}}(\gamma_{\vec{1}}x_{\vec{2}}x_{\vec{4}} + \gamma_{\vec{2}}x_{\vec{1}}x_{\vec{4}} - \gamma_{\vec{2}}x_{\vec{2}} - \gamma_{\vec{1}}x_{\vec{1}}) - \tfrac{1}{2}\gamma_{\vec{2}}^{-1}\epsilon_{\vec{2}}\vec{v}_{\vec{2}}\cdot\vec{k}_3(x_{\vec{2}} + x_{\vec{1}}x_{\vec{4}}) - \tfrac{1}{2}\gamma_{\vec{1}}^{-1}\epsilon_{\vec{1}}\vec{v}_{\vec{1}}\cdot\vec{k}_3(x_{\vec{1}} + x_{\vec{2}}x_{\vec{4}}) . \tag{A7}$$

Similarly, we find for Φ_{out}, the asymptotic form when an outgoing momentum $\vec{k}_1 \to 0$,

$$\Phi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = -\Phi^{(2)}_{\vec{2}\vec{1}\vec{3}\vec{4}} = (\epsilon_{\vec{2}} - \epsilon_{\vec{3}} - \epsilon_{\vec{4}})(1 - x_{\vec{2}}x_{\vec{3}}x_{\vec{4}}) , \tag{A8}$$

$$\Phi^{(2)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = -\Phi^{(8)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = (\epsilon_{\vec{2}} + \epsilon_{\vec{3}} + \epsilon_{\vec{4}})(x_{\vec{2}} - x_{\vec{3}}x_{\vec{4}}) , \tag{A9}$$

$$\Phi^{(3)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = -\Phi^{(4)}_{\vec{2}\vec{1}\vec{3}\vec{4}} = (\epsilon_{\vec{2}} + \epsilon_{\vec{4}} - \epsilon_{\vec{3}})(x_{\vec{2}}x_{\vec{3}} - x_{\vec{4}}) , \tag{A10}$$

$$\Phi^{(4)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = -\Phi^{(5)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = (\epsilon_{\vec{2}} + \epsilon_{\vec{3}} - \epsilon_{\vec{4}})(x_{\vec{3}} - x_{\vec{2}}x_{\vec{4}}) , \tag{A11}$$

$$\Phi^{(7)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = -\Phi^{(6)}_{\vec{2}\vec{1}\vec{3}\vec{4}} = (\epsilon_{\vec{2}} + \epsilon_{\vec{3}} + \epsilon_{\vec{4}})(x_{\vec{3}}x_{\vec{4}} - x_{\vec{2}}) , \tag{A12}$$

$$\Phi^{(6)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = -\Phi^{(9)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = (\epsilon_{\vec{3}} + \epsilon_{\vec{4}} - \epsilon_{\vec{2}})(1 - x_{\vec{2}}x_{\vec{3}}x_{\vec{4}}) . \tag{A13}$$

We shall also have occasion to use formulas when two momenta are small. In particular, we shall need these results for the treatment of the classical regime. We quote only those results (correct to linear order in the small momenta) which are actually needed for our calculations:

$$\Phi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim 2\epsilon_{\vec{2}}\epsilon_{\vec{3}}(1+\epsilon_{\vec{2}})^{-1}(\hat{k}_3\cdot\vec{v}_{\vec{2}} - 1) , \quad k_1, k_3 \to 0 \tag{A14}$$

$$\Phi^{(2)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim 2\epsilon_{\vec{1}}\epsilon_{\vec{3}}(1+\epsilon_{\vec{1}})^{-1}(1 - \hat{k}_3\cdot\vec{v}_{\vec{1}}) , \quad k_2, k_3 \to 0 \tag{A15a}$$

$$\Phi^{(2)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim 0 , \quad k_3, k_4 \to 0 \tag{A15b}$$

$$\Phi^{(3)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim 0 , \quad k_1, k_2 \to 0 \tag{A16a}$$

$$\Phi^{(3)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim -2\epsilon_{\vec{2}}\epsilon_{\vec{4}}(1+\epsilon_{\vec{2}})^{-1}(1 + \hat{k}_4\cdot\vec{v}_{\vec{2}}) , \quad k_1, k_4 \to 0 \tag{A16b}$$

$$\Phi^{(4)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim 0 , \quad k_1, k_2 \to 0 \tag{A17a}$$

$$\Phi^{(4)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim 2\epsilon_{\vec{2}}\epsilon_{\vec{3}}(1+\epsilon_{\vec{2}})^{-1}(1 + \hat{k}_3\cdot\vec{v}_{\vec{2}}) , \quad k_1, k_3 \to 0 \tag{A17b}$$

$$\Phi^{(4)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim 0 , \quad k_3, k_4 \to 0 \tag{A17c}$$

$$\Phi^{(5)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim -2\epsilon_{\vec{1}}\epsilon_{\vec{3}}(1+\epsilon_{\vec{1}})^{-1}(1 + \hat{k}_3\cdot\vec{v}_{\vec{1}}), \quad k_2, k_3 \to 0 \tag{A18}$$

$$\Phi^{(6)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim 2\epsilon_{\vec{2}}\epsilon_{\vec{4}}(1+\epsilon_{\vec{2}})^{-1}(1 - \hat{k}_4\cdot\vec{v}_{\vec{2}}), \quad k_1, k_4 \to 0 . \tag{A19}$$

In the quantum-mechanical regimes we are primarily interested in the case when all the momenta are small. Then we may use the following results from Ref. 62, which include the effects of anisotropy ($q_A = 2DS/H_E$):

$$\Phi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim 4q_A + (\tfrac{1}{2}\vec{k}_3\cdot\vec{k}_4 - 2\epsilon_{\vec{3}}\epsilon_{\vec{4}}) , \tag{A20a}$$

$$\Phi^{(2)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim -4q_A - (\tfrac{1}{2}\vec{k}_3\cdot\vec{k}_4 - 2\epsilon_{\vec{3}}\epsilon_{\vec{4}}) , \tag{A20b}$$

$$\Phi^{(3)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim -4q_A - (\tfrac{1}{2}\vec{k}_3\cdot\vec{k}_4 - 2\epsilon_{\vec{3}}\epsilon_{\vec{4}}) , \tag{A20c}$$

$$\Phi^{(4)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim 4q_A + (\tfrac{1}{2}\vec{k}_3\cdot\vec{k}_4 + 2\epsilon_{\vec{3}}\epsilon_{\vec{4}}) , \tag{A20d}$$

$$\Phi^{(5)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim 4q_A - (\tfrac{1}{2}\vec{k}_3\cdot\vec{k}_4 + 2\epsilon_{\vec{3}}\epsilon_{\vec{4}}) , \tag{A20e}$$

$$\Phi^{(6)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \sim -4q_A - (\tfrac{1}{2}\vec{k}_3\cdot\vec{k}_4 - 2\epsilon_{\vec{3}}\epsilon_{\vec{4}}) . \tag{A20f}$$

In the isotropic case these relations, together with the condition $\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4$, imply that

$$-\tfrac{1}{2}\Phi^{(6)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = \tfrac{1}{2}\Phi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = \epsilon_{\vec{3}}\epsilon_{\vec{4}}(\hat{k}_3\cdot\hat{k}_4 - 1) = \epsilon_{\vec{1}}\epsilon_{\vec{2}}(\hat{k}_1\cdot\hat{k}_2 - 1) + \tfrac{1}{2}\Delta\epsilon_{\vec{1}\vec{2}}(\epsilon_{\vec{1}} + \epsilon_{\vec{2}} + \epsilon_{\vec{3}} + \epsilon_{\vec{4}}) , \tag{A21a}$$

$$\tfrac{1}{2}\Phi^{(4)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = \epsilon_{\vec{3}}\epsilon_{\vec{4}}(\hat{k}_3\circ\hat{k}_4 + 1) = \epsilon_{\vec{1}}\epsilon_{\vec{2}}(\hat{k}_1\cdot\hat{k}_2 + 1) + \tfrac{1}{2}\Delta\epsilon_{\vec{1}\vec{3}}\Delta\epsilon_{\vec{1}\vec{4}} , \tag{A21b}$$

$$-\tfrac{1}{2}\Phi^{(3)}_{-\vec{1},\vec{4},\vec{2},-\vec{3}} = \epsilon_{\vec{2}}\epsilon_{\vec{3}}(1 - \hat{k}_3\cdot\hat{k}_2) = \epsilon_{\vec{1}}\epsilon_{\vec{4}}(1 - \hat{k}_1\cdot\hat{k}_4) + \tfrac{1}{2}\Delta\epsilon_{\vec{1}\vec{3}}\Delta\epsilon_{\vec{1}\vec{2}} , \tag{A21c}$$

$$-\tfrac{1}{2}\Phi^{(3)}_{-\vec{1},\vec{3},\vec{2},-\vec{4}} = \epsilon_{\vec{2}}\epsilon_{\vec{4}}(1 - \hat{k}_4\cdot\hat{k}_2) = \epsilon_{\vec{1}}\epsilon_{\vec{3}}(1 - \hat{k}_1\cdot\hat{k}_3) + \tfrac{1}{2}\Delta\epsilon_{\vec{1}\vec{4}}\Delta\epsilon_{\vec{1}\vec{2}} , \tag{A21d}$$

where we use the notation

$$\Delta\epsilon_{\vec{1}\vec{2}} = \epsilon_{\vec{1}} + \epsilon_{\vec{2}} - \epsilon_{\vec{3}} - \epsilon_{\vec{4}} , \tag{A22a}$$

$$\Delta\epsilon_{\vec{1}\vec{3}} = \epsilon_{\vec{1}} + \epsilon_{\vec{3}} - \epsilon_{\vec{2}} - \epsilon_{\vec{4}} , \tag{A22b}$$

and so forth. More generally, in both isotropic and anisotropic systems at long wavelengths we have the relations

$$\Phi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = \Phi^{(1)}_{\vec{3}\vec{4}\vec{1}\vec{2}} + \Delta\epsilon_{\vec{1}\vec{2}}(\epsilon_{\vec{1}} + \epsilon_{\vec{2}} + \epsilon_{\vec{3}} + \epsilon_{\vec{4}}) , \tag{A23a}$$

$$\Phi^{(3)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = \Phi^{(2)}_{\vec{3}\vec{4}\vec{1}\vec{2}} - (\epsilon_{\vec{3}} - \epsilon_{\vec{1}} - \epsilon_{\vec{2}} - \epsilon_{\vec{4}})(\epsilon_{\vec{4}} - \epsilon_{\vec{1}} - \epsilon_{\vec{2}} - \epsilon_{\vec{3}}) , \tag{A23b}$$

$$\Phi^{(4)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = \Phi^{(4)}_{\vec{3}\vec{4}\vec{1}\vec{2}} + \Delta\epsilon_{\vec{1}\vec{4}}\Delta\epsilon_{\vec{1}\vec{3}} , \tag{A23c}$$

$$\Phi^{(6)}_{\vec{1}\vec{2}\vec{3}\vec{4}} = \Phi^{(5)}_{\vec{3}\vec{4}\vec{1}\vec{2}} + (\epsilon_{\vec{2}} - \epsilon_{\vec{1}} - \epsilon_{\vec{3}} - \epsilon_{\vec{4}})(\epsilon_{\vec{1}} - \epsilon_{\vec{2}} - \epsilon_{\vec{3}} - \epsilon_{\vec{4}}) . \tag{A23d}$$

APPENDIX B: EVALUATION OF $\Sigma''_{\beta\alpha}(\vec{k}, \omega)$

Here we outline the evaluation of the off-diagonal elements of the mass operator $\Sigma''_{\beta\alpha}(\vec{k}, \omega)$. The diagrams which contribute to $\Sigma''_{\beta\alpha}(\vec{k}, \omega)$ in the lowest

Born approximation are the same as those of Figs. 1 and 2, except that the outgoing α magnon is now replaced by an incoming β magnon. Accordingly, the interaction coefficients associated with the outgoing vertex in the matrix elements M_{22} and M_{31} used in Sec. IV must be modified in order to evaluate $\Sigma''_{\beta\alpha}(\vec{k}, \omega)$. For instance, in $M_{22}(\vec{k}, \vec{p}, \vec{p}-\vec{q}, \vec{k}+\vec{q})$ one should made the replacements[70]

$$\Phi^{(1)}(\vec{k}, \vec{p}, \vec{p}-\vec{q}, \vec{k}+\vec{q}) \to \Phi^{(2)}(\vec{p}, \vec{k}, \vec{p}-\vec{q}, \vec{k}+\vec{q}) , \tag{B1a}$$

$$\Phi^{(4)}(\vec{k}, -\vec{k}-\vec{q}, \vec{p}-\vec{q}, -\vec{p}) \to \Phi^{(5)}(\vec{k}, -\vec{k}-\vec{q}, \vec{p}-\vec{q}, -\vec{p}) , \tag{B1b}$$

$$\Phi^{(4)}(\vec{k}, \vec{q}-\vec{p}, \vec{k}+\vec{q}, -\vec{p}) \to \Phi^{(5)}(\vec{k}, \vec{q}-\vec{p}, \vec{k}+\vec{q}, -\vec{p}) , \tag{B1c}$$

and in $M_{31}(\vec{k}, \vec{p}, \vec{p}-\vec{q}, \vec{k}+\vec{q})$ the replacements

$$\Phi^{(6)}(\vec{k}, \vec{p}, \vec{p}-\vec{q}, \vec{k}+\vec{q}) \to \Phi^{(9)}(\vec{k}, \vec{p}, \vec{p}-\vec{q}, \vec{k}+\vec{q}) , \tag{B1d}$$

$$\Phi^{(3)}(-\vec{k}, \vec{k}+\vec{q}, \vec{p}, \vec{q}-\vec{p}) \to \Phi^{(4)}(\vec{k}+\vec{q}, -\vec{k}, \vec{p}, \vec{q}-\vec{p}) , \tag{B1e}$$

$$\Phi^{(3)}(-\vec{k}, \vec{p}-\vec{q}, \vec{p}, -\vec{k}-\vec{q}) \to \Phi^{(4)}(\vec{p}-\vec{q}, -\vec{k}, \vec{p}, -\vec{k}-\vec{q}) . \tag{B1f}$$

Referring to Appendix A for the asymptotic forms of the interaction coefficients for small k, we see that the coefficients for $\Sigma_{\beta\alpha}$ differ from those for $\Sigma_{\alpha\alpha}$ by a factor -1. In fact, this result is an explicit example of

$$\Phi_{\rm out}(\vec{k}, \vec{k}_2, \vec{k}_3, \vec{k}_4) = -\Phi'_{\rm out}(\vec{k}, \vec{k}_2, \vec{k}_3, \vec{k}_4) \tag{B2}$$

in the sense of Eq. (G23). Consequently, we have

$$\Sigma''_{\beta\alpha}(\vec{k}, \omega) = -\Sigma''_{\alpha\alpha}(\vec{k}, \omega) = -\tfrac{1}{2} H_E (2\pi)^{-3} S^{-2} I_A\, \tilde{\omega}\epsilon_{\vec{k}}(1+3\rho) . \tag{B3}$$

Moreover, using the general symmetry relations, we have that

$$\Sigma''_{\alpha\beta}(\vec{k}, \omega) = -\Sigma''_{\beta\alpha}(\vec{k}, -\omega) = -\tfrac{1}{2} H_E (2\pi)^{-3} S^{-2} I_A\, \tilde{\omega}\epsilon_{\vec{k}}(1-3\rho) . \tag{B4}$$

The results for the classical case follow from Eqs. (B2) and (6.19):

$$\hbar^{-1}\Sigma''_{\beta\alpha}(\vec{k}, \omega) = -\hbar^{-1}\Sigma''_{\alpha\alpha}(\vec{k}, \omega) = -(4/3\pi)\,\omega_E\,(T/T_0)^2\,\tilde{\omega}\epsilon_{\vec{k}}\,(\tfrac{1}{4}\eta_2 + \tfrac{3}{4}\eta_0\rho) . \tag{B5}$$

Use of the general symmetry relations then leads to

$$\hbar^{-1}\Sigma''_{\alpha\beta}(\vec{k}, \omega) = -\hbar^{-1}\Sigma''_{\beta\alpha}(\vec{k}, -\omega) = -(4/3\pi)\,\omega_E\,(T/T_0)^2\,\tilde{\omega}\epsilon_{\vec{k}}\,(\tfrac{1}{4}\eta_2 - \tfrac{3}{4}\eta_0\rho) . \tag{B6}$$

APPENDIX C: EVALUATION OF THE VERTEX FUNCTION $\Lambda_{\mu\nu}(\vec{k},z)$

In this Appendix we shall describe the diagrammatic calculation of the vertex function $\Lambda_{\mu\nu}(\vec{k}, z)$ required for the evaluation of the spin-correlation functions in Sec. VII. The diagrammatic series for $\Lambda_{\mu\nu}(\vec{k}, z)$ is most conveniently represented as that for the self-energy $\Sigma_{\mu\nu}(\vec{k}, z)$ except that one external vertex is replaced by a suitably constructed external potential $V_{\rm eff}$. Thus we shall be able to use the standard rules for the diagrammatic calculation of $\Sigma^{\rm eff}_{\mu\nu}(\vec{k}, z)$ in the presence of a fictitious interaction.

1. Diagrammatic Series for $\Lambda_{\mu\nu}(\vec{k}, z)$

For convenience, let us first construct $\Lambda_{mn}(\vec{k}, z)$, or equivalently, the function $\mathcal{F}_{mn}(\vec{k}, z)$, which is defined as[68]

$$\mathcal{F}_{mn}(\vec{k}, z) = \mathcal{G}_{mn}(\vec{k}, z) - G_{mn}(\vec{k}, z) , \tag{C1}$$

where $\mathcal{G}$ and G are the spin and boson Green's functions defined in Eqs. (2.30) and (2.20). Explicitly, Eq. (C1) is

$$\mathcal{F}_{an}(\vec{k}, z) = \langle\langle (2S)^{-1/2} S^+_a(\vec{k}) - a_{\vec{k}} ;\ (2S)^{-1/2} S^-_n(\vec{k}) \rangle\rangle \tag{C2a}$$

$$= \langle\langle A_{\vec{k}} ; (2S)^{-1/2} S^-_n(\vec{k}) \rangle\rangle \tag{C2b}$$

$$\mathcal{F}_{bn}(k, z) = \langle\langle (2S)^{-1/2}(\vec{k}) - b^\dagger_{-\vec{k}} ;\ (2S)^{-1/2} S^-_n(\vec{k}) \rangle\rangle \tag{C3a}$$

$$= \langle\langle B^\dagger_{-\vec{k}} ; (2S)^{-1/2} S^+_b(\vec{k}) \rangle\rangle , \tag{C3b}$$

where $\langle\langle A ; B \rangle\rangle$ is the Fourier transform, as in Eq. (2.21), of the imaginary time Green's function $-i\langle T A(t) B(0) \rangle$. (Recall our convention with regard to Latin and Greek subscripts, Ref. 68.) It is clear that $\mathcal{F}_{\mu\nu}(\vec{k}, z)$ is a Green's function whose diagrams have the structure shown in Fig. 8. The three external lines on the left-hand side of the upper diagram of Fig. 8 carry total momentum $\vec{k}$ and energy z as indicated by the fact that they end at a single vertex. For comparison, we also show in Fig. 8 a typical diagram for $(\underline{G}^0\underline{\Sigma}^{\rm eff}\,\underline{G})_{\mu\nu}$, where we use matrix notation

$$(\underline{G}^0\,\underline{\Sigma}^{\rm eff}\,\underline{G})_{\mu\nu} = \sum_{\sigma\rho} G^0_{\mu\sigma}(\vec{k}, z)\,\Sigma^{\rm eff}_{\sigma\rho}(\vec{k}, z)\,G_{\rho\nu}(\vec{k}, z) . \tag{C4}$$

Note that we "read" diagrams from right to left, i.e., $\underline{G}(\vec{r}, t; \vec{r}\,', t')$ is conventionally represented by a line *from* $(\vec{r}\,', t')$ *to* $(\vec{r}, t)$. It is clear from Fig. 8, that if we construct $V_{\rm eff}$ correctly, and use it only at the outgoing vertex, we may write

$$\mathcal{F}_{\mu\nu}(\vec{k}, z) = \textstyle\sum_\rho \Sigma^{\rm eff}_{\mu\rho}(\vec{k}, z)\,G_{\rho\nu}(\vec{k}, z) . \tag{C5}$$

Consider the choice

$$V_{\rm eff} = \textstyle\sum_{\vec{k}} (l_{\vec{k}}\,\alpha^\dagger_{\vec{k}} - m_{\vec{k}}\,\beta_{\vec{k}})\,A_{\vec{k}} + \sum_{\vec{k}} (l_{\vec{k}}\,\beta_{-\vec{k}} - m_{\vec{k}}\,\alpha^\dagger_{\vec{k}})\,B^\dagger_{-\vec{k}} , \tag{C6}$$

where we require the external lines to be contracted only with the $\alpha^\dagger_{\vec{k}}$ or $\beta_{-\vec{k}}$ operators. We assert that this choice for V_{eff} satisfies Eq. (C5). To see this consider $(\underline{\Sigma}^{\text{eff}}\,\underline{G})_{\alpha\nu}$. From Fig. 8, and using the above expression for V_{eff}, we see that

$$(\underline{\Sigma}^{\text{eff}}\,\underline{G})_{\alpha\nu} = \langle\langle\, l_{\vec{k}} A_{\vec{k}} - m_{\vec{k}} B^\dagger_{-\vec{k}}\,;(2S)^{-1/2}S^+_\nu(\vec{k})\rangle\rangle \tag{C7a}$$

$$= \mathcal{F}_{\alpha\nu}(\vec{k},\ z)\ . \tag{C7b}$$

To obtain this equation we note that the external α line contracts only with the $\alpha^\dagger_{\vec{k}}$ operators of Eq. (C6). Similarly, we find

$$(\underline{\Sigma}^{\text{eff}}\,\underline{G})_{\beta\nu} = \langle\langle\, l_{\vec{k}} B^\dagger_{-\vec{k}} - m_{\vec{k}} A_{\vec{k}}\,;(2S)^{-1/2}S^+_\nu(\vec{k})\rangle\rangle \tag{C8a}$$

$$= \mathcal{F}_{\beta\nu}(\vec{k},\ z)\ , \tag{C8b}$$

so that indeed

$$(\underline{\Sigma}^{\text{eff}}\,\underline{G})_{\mu\nu} = \mathcal{G}_{\mu\nu} - G_{\mu\nu}\ , \tag{C9a}$$

$$(\underline{\Sigma}^{\text{eff}}\,\underline{G})_{\mu\nu} = (\underline{\Lambda}\underline{G})_{\mu\nu}\ , \tag{C9b}$$

which implies that

$$\Lambda_{\mu\nu}(\vec{k},\ z) = \Sigma^{\text{eff}}_{\mu\nu}(\vec{k},\ z)\ . \tag{C10}$$

It is apparent that we shall need V_{eff} in terms of the α's and β's. We write[61]

$$\begin{aligned}
-2S\,V_{\text{eff}} = N^{-1}\sum_{\vec{k}\vec{2}\vec{3}\vec{4}} l_{\vec{k}}\, l_{\vec{2}}\, l_{\vec{3}}\, l_{\vec{4}}\,\big(&\hat{\Phi}^{(1)}_{\vec{k}\vec{2}\vec{3}\vec{4}}\,\alpha^\dagger_{\vec{k}}\,\alpha^\dagger_{\vec{2}}\,\alpha_{\vec{3}}\,\alpha_{\vec{4}} \\
&+\hat{\Phi}^{(2)}_{\vec{k}\vec{2}\vec{3}\vec{4}}\,\alpha^\dagger_{\vec{k}}\,\beta_{-\vec{2}}\,\alpha_{\vec{3}}\,\alpha_{\vec{4}} - \hat{\Phi}^{(2)}_{\vec{2}\vec{k}\vec{3}\vec{4}}\,\beta_{-\vec{k}}\,\alpha^\dagger_{\vec{2}}\,\alpha_{\vec{3}}\,\alpha_{\vec{4}} \\
&+2\hat{\Phi}^{(3)}_{\vec{k}\vec{2}\vec{3}\vec{4}}\,\alpha^\dagger_{\vec{k}}\,\alpha^\dagger_{\vec{2}}\,\alpha_{\vec{3}}\,\beta^\dagger_{-\vec{4}} + 2\hat{\Phi}^{(5)}_{\vec{k}\vec{2}\vec{3}\vec{4}}\,\beta_{-\vec{k}}\,\beta_{-\vec{2}}\,\alpha_{\vec{3}}\,\beta^\dagger_{-\vec{4}} \\
&+2\hat{\Phi}^{(4)}_{\vec{k}\vec{2}\vec{3}\vec{4}}\,\alpha^\dagger_{\vec{k}}\,\beta_{-\vec{2}}\,\alpha_{\vec{3}}\,\beta^\dagger_{-\vec{4}} - 2\hat{\Phi}^{(4)}_{\vec{2}\vec{k}\vec{3}\vec{4}}\,\beta_{-\vec{k}}\,\alpha^\dagger_{\vec{2}}\,\alpha_{\vec{3}}\,\beta^\dagger_{-\vec{4}} \\
&+\hat{\Phi}^{(6)}_{\vec{k}\vec{2}\vec{3}\vec{4}}\,\alpha^\dagger_{\vec{k}}\,\beta_{-\vec{2}}\,\beta^\dagger_{-\vec{3}}\,\beta^\dagger_{-\vec{4}} - \hat{\Phi}^{(6)}_{\vec{2}\vec{k}\vec{3}\vec{4}}\,\beta_{-\vec{k}}\,\alpha^\dagger_{\vec{2}}\,\beta^\dagger_{-\vec{3}}\,\beta^\dagger_{-\vec{4}} \\
&+\hat{\Phi}^{(7)}_{\vec{k}\vec{2}\vec{3}\vec{4}}\,\alpha^\dagger_{\vec{k}}\,\alpha^\dagger_{\vec{2}}\,\beta^\dagger_{-\vec{3}}\,\beta^\dagger_{-\vec{4}} + \hat{\Phi}^{(8)}_{\vec{k}\vec{2}\vec{3}\vec{4}}\,\beta_{-\vec{k}}\,\beta_{-\vec{2}}\,\alpha_{\vec{3}}\,\alpha_{\vec{4}} \\
&+\hat{\Phi}^{(9)}_{\vec{k}\vec{2}\vec{3}\vec{4}}\,\beta_{-\vec{k}}\,\beta_{-\vec{2}}\,\beta^\dagger_{-\vec{3}}\,\beta^\dagger_{-\vec{4}}\big)\ ,
\end{aligned} \tag{C11}$$

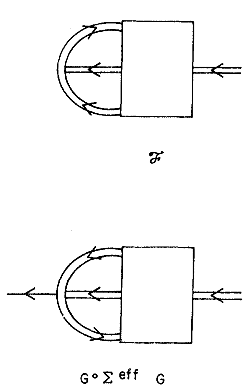

FIG. 8. Equivalence of the vertex function $\underline{\Lambda}$ to an effective self-energy. The upper diagram shows the structure of $\underline{\mathcal{F}}$, that part of the spin Green's function arising out of the cubic terms in the Dyson-Maleev representation of the spin operators. The lower diagram shows the structure of $\underline{G}^0\Sigma^{\text{eff}}\underline{G}$ for a boson system with the usual quartic interactions at the internal vertices, and with an effective interaction on the left external vertex. Here double lines represent true boson propagators and single lines the unperturbed propagators.

where the coefficients are determined from the definition, Eq. (C6), as

$$\hat{\Phi}^{(1)}_{\vec{k}\vec{2}\vec{3}\vec{4}} = \hat{\Phi}^{(9)}_{\vec{k}\vec{2}\vec{3}\vec{4}} = 1 - x_{\vec{k}}\,x_{\vec{2}}\,x_{\vec{3}}\,x_{\vec{4}}\ , \tag{C12a}$$

$$\hat{\Phi}^{(2)}_{\vec{k}\vec{2}\vec{3}\vec{4}} = \hat{\Phi}^{(3)}_{\vec{k}\vec{2}\vec{3}\vec{4}} = \hat{\Phi}^{(5)}_{\vec{k}\vec{2}\vec{3}\vec{4}} = -\,\hat{\Phi}^{(6)}_{\vec{2}\vec{k}\vec{3}\vec{4}} = -\,x_{\vec{2}} + x_{\vec{k}}\,x_{\vec{3}}\,x_{\vec{4}}\ , \tag{C12b}$$

$$\hat{\Phi}^{(4)}_{\vec{k}\vec{2}\vec{3}\vec{4}} = \hat{\Phi}^{(7)}_{\vec{k}\vec{3}\vec{2}\vec{4}} = \hat{\Phi}^{(8)}_{\vec{k}\vec{3}\vec{2}\vec{4}} = x_{\vec{2}}\,x_{\vec{4}} - x_{\vec{k}}\,x_{\vec{3}}\ . \tag{C12c}$$

To summarize: $\Lambda_{\mu\nu}(\vec{k},\ z)$ is calculated according to the usual rules for a self-energy diagram, except that the potential at the outgoing vertex is taken to be that of Eq. (C11) with the proviso that the external line contracts only with the $\alpha^\dagger_{\vec{k}}$ or $\beta_{-\vec{k}}$ operators.

2. Calculation of $\Lambda''_{\mu\nu}(\vec{k},\omega)$ in Lowest Born Approximation

We shall now evaluate $\Lambda''_{\mu\nu}(\vec{k},\omega)$ in the lowest Born approximation. We shall initially consider the quantum low-temperature limit, regime A, $\epsilon_{\vec{k}} \ll \tau^3$. From our previous discussion, we see that the calculation of $\Lambda''_{\alpha\alpha}(\mathrm{k},\ z)$, for instance, is similar to that of $\Sigma''_{\alpha\alpha}(\mathrm{k},\ z)$, except that the coefficients $\Phi^{(i)}$ are replaced by $\hat{\Phi}^{(i)}$ at the outgoing vertex. Denoting the corresponding matrix elements[106] as M'_{22} and M'_{31} we have, for $k/p \ll 1$ and $q/p \ll 1$,

$$M'_{22}(\vec{k},\vec{p};\vec{k}+\vec{q},\vec{p}-\vec{q}) = (4/\epsilon_{\vec{s}})\,(\mu-1)\ , \tag{C13a}$$

$$M'_{31}(\vec{k},\vec{p};\vec{k}+\vec{q},\vec{p}-\vec{q}) = (4/\epsilon_{\vec{s}})\,(-\,\mu-1)\ . \tag{C13b}$$

Thus we find the analog of Eq. (4.8):

$$\begin{aligned}
\Lambda''_{\alpha\alpha}(\vec{k},\omega) &= 2S^{-2}(2\pi)^{-3}\rho\,\epsilon_{\vec{k}}\hat{K}(\epsilon_{\vec{k}},\rho)\ ,\\
\hat{K}(\epsilon_{\vec{k}},\rho) &= (2\pi)^{-2}\int d\vec{p}\int d\vec{s}\,n_{\vec{p}}(1+n_{\vec{p}})s^{-2}\\
&\quad\times[(\mu-1)\,\delta(2\epsilon_{\vec{k}}\rho+2\Delta)\\
&\quad-(\mu+1)\,\delta(-2\epsilon_{\vec{k}}\rho+2\Delta)]\ .
\end{aligned} \tag{C14}$$

This integral can be done in exactly the manner of Appendix D, and so we quote only the result[107]:

$$\hat{K}(\epsilon_{\vec{k}},\rho) = -\tfrac{3}{4}I_A\ , \tag{C15}$$

where I_A is again given by Eq. (3.27). Thus

$$\Lambda''_{\alpha\alpha}(\vec{k},\omega) = -\tfrac{3}{2}S^{-2}(2\pi)^{-3}I_A\tilde{\omega}\ . \tag{C16}$$

To obtain $\Lambda''_{\beta\beta}(\vec{k},\omega)$, we use the symmetry relation

$$\Lambda''_{\beta\beta}(\vec{k},\omega) = -\,\Lambda''_{\alpha\alpha}(\vec{k},-\omega)\ , \tag{C17}$$

so that from our above result for $\Lambda''_{\alpha\alpha}(\vec{k},\omega)$ we have

$$\Lambda''_{\beta\beta}(\vec{k},\omega) = \Lambda''_{\alpha\alpha}(\vec{k},\omega)\ . \tag{C18}$$

Next we evaluate $\Lambda''_{\beta\alpha}(\vec{k}, \omega)$. Here the calculation is analogous to that for $\Sigma''_{\beta\alpha}(\vec{k}, \omega)$ given in Appendix B, except for the replacement of $\Phi^{(i)}$ by $\hat{\Phi}^{(i)}$ at the outgoing vertex. Denoting the corresponding matrix elements by N'_{22} and N'_{31}, we have for $k/p \ll 1$ and $q/p \ll 1$,

$$N'_{22}(\vec{k}, \vec{p}; \vec{k}+\vec{q}, \vec{p}-\vec{q}) = M'_{22}(\vec{k}, \vec{p}; \vec{k}+\vec{q}, \vec{p}-\vec{q}) , \quad \text{(C19a)}$$

$$N'_{31}(\vec{k}, \vec{p}; \vec{k}+\vec{q}, \vec{p}-\vec{q}) = M'_{31}(\vec{k}, \vec{p}; \vec{k}+\vec{q}, \vec{p}-\vec{q}) , \quad \text{(C19b)}$$

where we have used[108]

$$\hat{\Phi}_{\text{out}} = \hat{\Phi}'_{\text{out}} \quad \text{for} \quad k \to 0 , \quad \text{(C20)}$$

i.e.,

$$\hat{\Phi}^{(1)}(\vec{k}, \vec{p}; \vec{k}+\vec{q}, \vec{p}-\vec{q}) = -\hat{\Phi}^{(2)}(\vec{p}, \vec{k}; \vec{k}+\vec{q}, \vec{p}-\vec{q}) \quad \text{for} \quad k \to 0 , \quad \text{(C21)}$$

and so forth. Equations (C19a) and (C19b) imply that

$$\Lambda''_{\beta\alpha}(\vec{k}, \omega) = \Lambda''_{\alpha\alpha}(\vec{k}, \omega) . \quad \text{(C22)}$$

Furthermore, use of the general symmetry relations given in Eq. (2.24) yields

$$\Lambda''_{\alpha\beta}(\vec{k}, \omega) = -\Lambda''_{\beta\alpha}(\vec{k}, -\omega) = \Lambda''_{\alpha\alpha}(\vec{k}, \omega) . \quad \text{(C23)}$$

We may summarize our results as

$$\Lambda''_{\mu\nu}(\vec{k}, \omega) = -\tfrac{3}{2} S^{-2} (2\pi)^{-3} I_A \tilde{\omega} . \quad \text{(C24)}$$

3. Classical Regime

Again the calculations for $\Lambda''_{\mu\nu}$ are quite similar to those for $\Sigma''_{\mu\nu}(\vec{k}, z)$ presented in Sec. VI. In particular, $\Lambda''_{\mu\nu}(\vec{k}, \omega)$ is given by the analog of Eq. (C14), except that we must use the classical analogs of the matrix elements M'_{22}, M'_{31}, N'_{22}, and N'_{31} introduced in Appendix C 2. As discussed in Sec. VI, we may assume $k/p \ll 1$ and $q/p \ll 1$, but p ranges over the entire zone. The expansions for $\hat{\Phi}^{(i)}$ are very simple, e.g.,

$$\hat{\Phi}^{(1)}(\vec{k}, \vec{p}; \vec{k}+\vec{q}, \vec{p}-\vec{q}) \approx 1 - x_{\vec{p}}^2 = 2\epsilon_{\vec{p}} / (1+\epsilon_{\vec{p}}) , \quad \text{(C25)}$$

and so forth. As a result, we find in the regime described

$$M'_{22}(\vec{k}, \vec{p}; \vec{k}+\vec{q}, \vec{p}-\vec{q}) = (4/\epsilon_{\vec{k}+\vec{q}})(\hat{k} \cdot \vec{v}_{\vec{p}} - 1) , \quad \text{(C26a)}$$

$$M'_{31}(\vec{k}, \vec{p}; \vec{k}+\vec{q}, \vec{p}-\vec{q}) = (4/\epsilon_{\vec{k}+\vec{q}})(-\hat{k} \cdot \vec{v}_{\vec{p}} - 1) . \quad \text{(C26b)}$$

To evaluate $\Lambda''_{\alpha\alpha}(\vec{k}, \omega)$ we insert these matrix elements into Eq. (6.18). Then we obtain the result

$$\Lambda''_{\alpha\alpha}(\vec{k}, \omega) = -(\eta_0/\pi)(T/T_0)^2 \tilde{\omega} . \quad \text{(C27)}$$

To calculate $\Lambda''_{\beta\alpha}(\vec{k}, \omega)$ we proceed as in Appendix C 2. Replacing $\hat{\Phi}^{(1)}(\vec{k}, \vec{p}; \vec{k}+\vec{q}, \vec{p}-\vec{q})$ by $\hat{\Phi}^{(2)}(\vec{p}, \vec{k}; \vec{k}+\vec{q}, \vec{p}-\vec{q})$ and so forth, as is required for the matrix elements N'_{22} and N'_{31} for $\Lambda''_{\beta\alpha}(\vec{k}, \omega)$, we find

$$N'_{22} = M'_{22} , \quad \text{(C28a)}$$

$$N'_{31} = M'_{31} , \quad \text{(C28b)}$$

so that again all the $\Lambda''_{\mu\nu}(\vec{k}, \omega)$ are the same. Thus in the low-temperature classical regime we have

$$\Lambda''_{\mu\nu}(\vec{k}, \omega) = -(\eta_0/\pi)(T/T_0)^2 \tilde{\omega} . \quad \text{(C29)}$$

4. Self-Consistency

We now consider the effect of requiring self-consistency in the preceding calculations. The discussion can be based on Eq. (D15). The present calculations differ from those of Appendix D only in the form for $F(y, \mu)$. In the present calculation we have

$$F(y, \mu) \propto 1 - \mu . \quad \text{(C30)}$$

For this form of $F(y, \mu)$, the discussion in Appendix D 3 shows that the result for $\Lambda''_{\mu\nu}(\vec{k}, \omega)$ is stable when damping of intermediate states is taken into account. These arguments apply equally well to the low-temperature quantum and classical regimes.

APPENDIX D: EVALUATION OF INTEGRALS IN SEC. IV

1. Evaluation of Eq. (4.9)

We wish to evaluate the integral in Eq. (4.9), which we write as

$$K = K_+ + K_- , \quad \text{(D1)}$$

where

$$K_+ = (2\pi)^{-2} \int d\vec{p} \int d\vec{s}\, n_{\vec{p}}(1+n_{\vec{p}}) s^{-2} \delta(2\epsilon_{\vec{k}}\rho + 2\Delta) \times [(1-\mu)^2 - (1-\rho)(1-\mu)] , \quad \text{(D2a)}$$

$$K_- = (2\pi)^{-2} \int d\vec{p} \int d\vec{s}\, n_{\vec{p}}(1+n_{\vec{p}}) s^{-2} \delta(-2\epsilon_{\vec{k}}\rho + 2\Delta) \times [(1+\mu)^2 - (1-\rho)(1+\mu)] . \quad \text{(D2b)}$$

Proceeding as in Sec. III B, we may use the δ function to do the ν integral which yields limits on s given by

$$s_+ \equiv \frac{(\rho - v_{\vec{p}}\mu)k}{1+v_{\vec{p}}} \le s \le \frac{(\rho - v_{\vec{p}}\mu)k}{1-v_{\vec{p}}} \equiv s'_+ , \quad \text{(D3a)}$$

$$s_- \equiv \frac{(-\rho - v_{\vec{p}}\mu)k}{1+v_{\vec{p}}} \le s \le \frac{(-\rho - v_{\vec{p}}\mu)k}{1-v_{\vec{p}}} \equiv s'_- , \quad \text{(D3b)}$$

for K_+ and K_-, respectively. Since we must have $s \ge 0$, the μ integral is limited by the conditions

$$-1 \le \mu \le \mu^+ \quad \text{for} \quad K_+ , \quad \text{(D4a)}$$

$$-1 \le \mu \le -\mu^+ \quad \text{for} \quad K_- , \quad \text{(D4b)}$$

with $\mu^+ = \min\{1, \rho/v_p\}$. Equations (D2) then become

$$K_+ = \int_0^\infty dp\, p^2 (1+n_p) n_p v_p^{-1} \int_{-1}^{\mu^+} d\mu \times \int_{s_+}^{s'_+} \frac{ds}{s} [(1-\mu)^2 - (1-\rho)(1-\mu)] , \quad \text{(D5a)}$$

$$K_-=\int_0^\infty dp\,p^2(1+n_p)n_p v_p^{-1}\int_{-1}^{-\mu^+}d\mu\times\int_{s_-}^{s'_-}\frac{ds}{s}\left[(1+\mu)^2-(1-\rho)(1+\mu)\right]\ ,\tag{D5b}$$

where $v_p=1-3g_{av}p^2$ [see Eq. (3.29)]. We may now do the s integrals, at which point k drops out completely, so that the limit $k\to 0$ is trivial. Carrying out the remaining integrals, we find

$$K=K_++K_-=\tfrac{1}{4}(1+3\rho)I_A\ ,\tag{D6}$$

where I_A is given by Eq. (3.27).

2. Evaluation of Eq. (4.21) for $(\epsilon_{\vec{k}}/\gamma)\ll 1$

Let us first evaluate Eq. (4.21),

$$\tilde{K}(\epsilon_{\vec{k}},\rho)=(2\pi)^{-2}\int d\vec{p}\int d\vec{s}\,n_{\vec{p}}(1+n_{\vec{p}})s^{-2}\int_{-\infty}^{+\infty}d\alpha\,(1-\mu)\Delta^0\delta(2\Delta^0-\alpha)\times\{\epsilon_{\vec{k}}^{-1}[\varphi(\alpha-2\epsilon_{\vec{k}}\rho+2\epsilon_{\vec{k}}\mu)-\varphi(\alpha+2\epsilon_{\vec{k}}\rho-2\epsilon_{\vec{k}}\mu)]\}\ ,\tag{4.21}$$

in the limit $\epsilon_{\vec{k}}/\gamma\ll 1$, where the curly bracket in Eq. (4.21) may be expanded in powers of $\epsilon_{\vec{k}}$. Let us set

$$\varphi(\alpha)=\gamma^{-1}f(\alpha/\gamma)\ ,\tag{D7}$$

where γ is the thermal width defined by Eq. (4.17), and f is a numerical function of order unity. We then have

$$\epsilon_{\vec{k}}^{-1}[\varphi(\alpha-2\rho\epsilon_{\vec{k}}+2\epsilon_{\vec{k}}\mu)-\varphi(\alpha+2\rho\epsilon_{\vec{k}}-2\epsilon_{\vec{k}}\mu)]\approx 4\gamma^{-2}\left[(\mu-\rho)f'\left(\frac{\alpha}{\gamma}\right)+\frac{2}{3}(\mu-\rho)^3f'''\left(\frac{\alpha}{\gamma}\right)\left(\frac{\epsilon_{\vec{k}}}{\gamma}\right)^2+\cdots\right]\ ,\tag{D8}$$

so that the expansion is indeed in the small parameter $(\epsilon_{\vec{k}}/\gamma)^2$, and the series may be terminated [note that the derivatives $f'(\alpha/\gamma)$, $f'''(\alpha/\gamma)$, etc., are of order unity]. Equation (4.21) then has the form

$$\tilde{K}(\epsilon_{\vec{k}},\rho)=(2\pi)^{-2}\int d\vec{p}\int d\vec{s}\,n_p(1+n_p)s^{-2}\int_{-\infty}^{\infty}dx\,\Delta^0(1-\mu)\,\delta(2\Delta^0-\gamma x)4\gamma^{-1}f'(x)\ ,\tag{D9}$$

where $2\Delta^0=-s+v_{\vec{p}}s\nu$. Since $\tilde{K}$ is independent of $\vec{k}$, the limit $\epsilon_{\vec{k}}\to 0$ is trivial. Proceeding as in Sec. III, we use the δ function to do the ν integral, which yields the limits

$$\gamma x(1+v_{\vec{p}})^{-1}\le s\le\gamma x(1-v_{\vec{p}})^{-1}\ ,\tag{D10a}$$

$$x>0\ .\tag{D10b}$$

Performing the s and μ integrals we find

$$\tilde{K}(0,\rho)=-\tfrac{4}{3}(1+3\rho)\int_0^\infty dp\,n_p(1+n_p)p^2\ln\left(\frac{1+v_p}{1-v_p}\right)\int_0^\infty xf'(x)\,dx\ .\tag{D11}$$

Using Eq. (4.18) and the fact that φ is an even function we have

$$\int_0^\infty xf'(x)\,dx=-\int_0^\infty\varphi(\alpha)\,d\alpha=-\tfrac{1}{2}\ ,\tag{D12}$$

which yields Eq. (4.21).

3. Evaluation of Eq. (4.21) to All Orders in $\epsilon_{\vec{k}}/\gamma$

Let us break up Eq. (4.21) into two parts as in Eq. (4.9):

$$\tilde{K}_+=(2\pi)^{-2}\int d\vec{p}\int d\vec{s}\,n_{\vec{p}}(1+n_{\vec{p}})s^{-2}\int_{-\infty}^{+\infty}\varphi(\alpha)\,d\alpha\,\delta(2\epsilon_{\vec{k}}\rho+2\Delta_+-\alpha)[(1-\mu)^2-(1-\mu)(1-\rho+\alpha/2\epsilon_{\vec{k}})]\ ,\tag{D13a}$$

$$\tilde{K}_-=(2\pi)^{-2}\int d\vec{p}\int d\vec{s}\,n_{\vec{p}}(1+n_{\vec{p}})s^{-2}\int_{-\infty}^{+\infty}\varphi(\alpha)\,d\alpha\,\delta(-2\epsilon_{\vec{k}}\rho+2\Delta_+-\alpha)[(1+\mu)^2-(1+\mu)(1-\rho-\alpha/2\epsilon_{\vec{k}})]\ ,\tag{D13b}$$

with $2\Delta_\pm=-s+v_{\vec{p}}s\nu\pm v_{\vec{p}}k\mu$. If μ is replaced by $-\mu$ in K_-, then these integrals are of the form

$$\tilde{K}_\pm=(2\pi)^{-2}\int d\vec{p}\int d\vec{s}\,n_{\vec{p}}(1+n_{\vec{p}})s^{-2}\int_{-\infty}^{+\infty}\varphi(\alpha)\,d\alpha\,\delta(\pm 2\epsilon_{\vec{k}}\rho+2\Delta_\pm-\alpha)F(\pm y,\mu)\ ,\tag{D14}$$

where we have set $\alpha = 2\epsilon_{\vec{k}} y$ and

$$F(y, \mu) = \mu^2 - (1+\rho - y)\mu + \rho - y \ . \tag{D15}$$

We shall carry out the calculation in such a way that results for different functions F which occur in the calculation of $\underline{\Sigma}$ or of $\underline{\Lambda}$ can be inferred immediately.

As before we use the δ function to do the ν integral, obtaining the limits

$$s_+ \equiv \frac{(\rho - y - v_{\vec{p}}\mu)k}{1+v_{\vec{p}}} \leq s \leq \frac{(\rho - y - v_{\vec{p}}\mu)k}{1 - v_{\vec{p}}} \equiv s_+' \ , \tag{D16a}$$

$$s_- \equiv \frac{(-\rho - y + v_{\vec{p}}\mu)k}{1+v_{\vec{p}}} \leq s \leq \frac{(-\rho - y + v_{\vec{p}}\mu)k}{1 - v_{\vec{p}}} \equiv s_-' \tag{D16b}$$

for $\tilde{K}_+$ and $\tilde{K}_-$, respectively. The condition $s \geq 0$ sets the limits on the μ integral

$$-1 \leq \mu \leq \mu^+(y) \quad \text{for} \quad \tilde{K}_+ , \tag{D17a}$$

$$\mu^+(-y) \leq \mu \leq 1 \quad \text{for} \quad \tilde{K}_- , \tag{D17b}$$

where

$$\mu^+(y) = -1 \ , \quad y \geq \rho + v_p \tag{D18a}$$

$$\mu^+(y) = v_p^{-1}(\rho - y) \ , \quad \rho + v_p \geq y \geq \rho - v_p \tag{D18b}$$

$$\mu^+(y) = 1 \ , \quad \rho - v_p \geq y \ . \tag{D18c}$$

Performing the s integrals, we obtain

$$\tilde{K} = \tilde{K}_+ + \tilde{K}_- = \int_0^\infty dp\, n_p(1+n_p)p^2 \ln\left(\frac{1+v_p}{1-v_p}\right) \times \int_{-\infty}^{+\infty} d\alpha\, \varphi(\alpha) G(\alpha, \rho) , \tag{D19}$$

with

$$G(2\epsilon_{\vec{k}} y, \rho) = \int_{-1}^{\mu^+(y)} F(y, \mu)\, d\mu + \int_{\mu^+(-y)}^{1} F(-y, \mu)\, d\mu \ . \tag{D20}$$

Since $\varphi(\alpha)$ is assumed to be an even function, we need keep only the even part of $G(\alpha)$, which we denote by $G_+(\alpha)$. From Eq. (D20), we find

$$G_+(2\epsilon_{\vec{k}} y, \rho) \equiv \tfrac{1}{2}[G(2\epsilon_{\vec{k}} y, \rho) + G(-2\epsilon_{\vec{k}} y, \rho)] \tag{D21a}$$

$$= \tfrac{1}{2}\int_{-1}^{1} d\mu [F(y, \mu) + F(-y, \mu)] \ . \tag{D21b}$$

In the present case, this result yields

$$G_+(\alpha, \rho) = \tfrac{2}{3} + 2\rho \ , \tag{D22}$$

and thus we find, using Eq. (4.18),

$$\tilde{K}(\epsilon_{\vec{k}}, \rho) = \tfrac{1}{4}(1 + 3\rho) I_A \ , \tag{D23}$$

so that $\tilde{K}$ is stable with respect to the inclusion of damping in the intermediate states.

More generally, we can say that an integral of the type given in Eq. (D19) is self-consistent if $G_+(\alpha, \rho)$ is independent of α. From Eq. (D21b) it is clear that this condition will be fulfilled if all terms in $F(y, \mu)$ which are even in both μ and y are independent of y, as is true, for instance, when F is given by Eq. (D15). In addition Eq. (D21b) justifies averaging $F(y, \mu)$ over μ and keeping only the part even in y.

APPENDIX E: STABILITY OF MAGNONS AGAINST SPONTANEOUS DECAY

In this Appendix we shall prove that the requirements of conservation of energy and momentum forbid the spontaneous decay of one magnon into two or more magnons, for a spin system where the only interactions are antiferromagnetic couplings between spins on opposite sublattices. The unperturbed spin-wave spectrum for this system is

$$E_{\vec{k}}^2 = J_{AB}^2(0) - J_{AB}^2(\vec{k}) \ , \tag{E1}$$

where

$$J_{AB}(\vec{k}) = \sum_{\vec{r}} J(\vec{r}) e^{i\vec{k}\cdot\vec{r}} \ , \tag{E2}$$

where the sum is over $\vec{r}$ in the sublattice including $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and $J(\vec{r})$ is the magnitude of the antiferromagnetic interaction between the spin at the origin and the one at $\vec{r}$. We write

$$E_{\vec{k}}^2 = \sum_{\vec{r}, \vec{s}} J(\vec{r}) J(\vec{s}) (1 - e^{i\vec{k}\cdot(\vec{r}+\vec{s})}) \tag{E3a}$$

$$= \tfrac{1}{2} \sum_{\vec{r}, \vec{s}} J(\vec{r}) J(\vec{s}) (2 - e^{i\vec{k}\cdot(\vec{r}+\vec{s})} - e^{-i\vec{k}\cdot(\vec{r}+\vec{s})}) \tag{E3b}$$

$$= 2 \sum_{\vec{r}, \vec{s}} J(\vec{r}) J(\vec{s}) \sin^2[\tfrac{1}{2}\vec{k}\cdot(\vec{r}+\vec{s})] \ , \tag{E3c}$$

which shows that $E_{\vec{k}}^2$ is of the form

$$E_{\vec{k}}^2 = \sum_{\vec{l}} \psi(\vec{l}) \sin^2(\tfrac{1}{2}\vec{k}\cdot\vec{l}) \ , \tag{E4}$$

where the coefficients $\psi(\vec{l})$ are non-negative, because

$$\psi(\vec{l}) = 2\sum_{\vec{r}} J(\vec{r}) J(\vec{l} - \vec{r}) \geq 0 \ . \tag{E5}$$

Accordingly, we may write

$$(E_{\vec{k}} + E_{\vec{p}})^2 = E_{\vec{k}}^2 + E_{\vec{p}}^2 + 2E_{\vec{k}}E_{\vec{p}} \ , \tag{E6a}$$

$$(E_{\vec{k}} + E_{\vec{p}})^2 = \sum_{\vec{l}} \psi(\vec{l}) [\sin^2(\tfrac{1}{2}\vec{k}\cdot\vec{l}) + \sin^2(\tfrac{1}{2}\vec{p}\cdot\vec{l})] + 2\left(\sum_{\vec{l}} \psi(\vec{l}) \sin^2(\tfrac{1}{2}\vec{k}\cdot\vec{l}) \sum_{\vec{l}'} \psi(\vec{l}') \sin^2(\tfrac{1}{2}\vec{p}\cdot\vec{l}')\right)^{1/2} . \tag{E6b}$$

But by the Cauchy-Schwartz inequality,

$$\left(\sum_{\vec{l}\vec{l}'} \psi(\vec{l}) \sin^2(\tfrac{1}{2}\vec{k}\cdot\vec{l}) \psi(\vec{l}') \sin^2(\tfrac{1}{2}\vec{p}\cdot\vec{l}')\right)^{1/2} \geq \sum_{\vec{l}} \psi(\vec{l}) |\sin(\tfrac{1}{2}\vec{k}\cdot\vec{l}) \sin(\tfrac{1}{2}\vec{p}\cdot\vec{l})| \ , \tag{E7}$$

so that

$$(E_{\vec{k}} + E_{\vec{p}})^2 \geq \sum_{\vec{l}} \psi(\vec{l}) [|\sin(\tfrac{1}{2}\vec{k}\cdot\vec{l})| + |\sin(\tfrac{1}{2}\vec{p}\cdot\vec{l})|]^2 \ . \tag{E8}$$

But

$$|\sin(\tfrac{1}{2}\vec{k}\cdot\vec{1})| + |\sin(\tfrac{1}{2}\vec{p}\cdot\vec{1})| \geq |\sin(\tfrac{1}{2}\vec{k}\cdot\vec{1})\cos(\tfrac{1}{2}\vec{p}\cdot\vec{1}) + \sin(\tfrac{1}{2}\vec{p}\cdot\vec{1})\cos(\tfrac{1}{2}\vec{k}\cdot\vec{1})| = |\sin(\tfrac{1}{2}(\vec{k}+\vec{p})\cdot\vec{1})| , \quad \text{(E9)}$$

so that

$$(E_{\vec{k}}+E_{\vec{p}})^2 \geq \sum \psi(\vec{1}) \sin^2(\tfrac{1}{2}(\vec{k}+\vec{p})\cdot\vec{1}) , \quad \text{(E10)}$$

and thus

$$E_{\vec{k}}+E_{\vec{p}} \geq E_{\vec{k}+\vec{p}} , \quad \text{(E11)}$$

where the equality is only realized if either k or p vanishes. As a corollary to this, we see that for any n

$$E_{\vec{k}_1}+E_{\vec{k}_2}+\cdots E_{\vec{k}_n} > E_{\vec{k}_1+\vec{k}_2+\cdots+\vec{k}_n} , \quad \text{(E12)}$$

except for the case where all $\vec{k}$'s but one are zero. It follows that one magnon cannot decay into two or more magnons, simultaneously conserving energy and momentum.

A second corollary of Eq. (E11) is that if r is sufficiently small, and $|\vec{k}+\vec{p}|$ is not close to zero, then the condition

$$E_{\vec{r}}+E_{\vec{k}+\vec{p}-\vec{r}} = E_{\vec{k}}+E_{\vec{p}} \quad \text{(E13)}$$

can only be satisfied if either k or p are close to zero. This theorem thus shows that there exists a regime in which the scattering surface defined in Sec. III A consists of disjoint pieces. The same conclusion also holds by continuity for k or $p=0$ for sufficiently small anisotropy.

Although the relations (E12) and (E13) were derived using the unrenormalized magnon energies, we expect these relations to hold also for the renormalized energies at low temperatures, at least for $(zS)^{-1} \ll 1$, since one finds $\mathrm{Re}\,\|\Sigma\|/E_{\vec{k}} \ll 1$.

APPENDIX F: CALCULATIONS USING HOLSTEIN-PRIMAKOFF FORMALISM

1. Formalism

In this Appendix we shall perform the calculation of Secs. III, IV, VI, and VII using the Holstein-Primakoff Hamiltonian[2] obtained from Eq. (2.1) via the transformation[109]

$$\tilde{S}_i^z = S - a_i^\dagger a_i , \quad \text{(F1a)}$$

$$\tilde{S}_i^+ = (2S)^{1/2}(1-f_i)^{1/2} a_i , \quad \text{(F1b)}$$

$$\tilde{S}_i^- = (\tilde{S}_i^+)^\dagger = (2S)^{1/2} a_i^\dagger (1-f_i)^{1/2} , \quad \text{(F1c)}$$

$$\tilde{S}_j^z = -S + b_j^\dagger b_j , \quad \text{(F1d)}$$

$$\tilde{S}_j^+ = (2S)^{1/2} b_j^\dagger (1-g_j)^{1/2} , \quad \text{(F1e)}$$

$$\tilde{S}_j^- = (\tilde{S}_j^+)^\dagger = (2S)^{1/2}(1-g_j)^{1/2} b_j , \quad \text{(F1f)}$$

where

$$f_i = (2S)^{-1} a_i^\dagger a_i , \quad \text{(F2a)}$$

$$g_j = (2S)^{-1} b_j^\dagger b_j . \quad \text{(F2b)}$$

Treating f_i and g_i as small parameters, we may expand the radicals appearing in the Hamiltonian obtained by inserting Eq. (F1) into Eq. (2.1). Performing the transformation to normal modes, Eq. (2.13), we then find

$$\mathcal{H}'_{\mathrm{HP}} = E'_0 + \mathcal{H}'_0 + V'_{\mathrm{HP}} , \quad \text{(F3)}$$

where E'_0 is a constant,[3,4] $\mathcal{H}'_0$ is again given by Eq. (2.17a), and

$$V'_{\mathrm{HP}} = V^{(0)}_{\mathrm{HP}} + V^{(1)}_{\mathrm{HP}} + \cdots , \quad \text{(F4)}$$

where $V^{(n)}_{\mathrm{HP}}$ is proportional to S^{-n}. Here

$$V^{(0)}_{\mathrm{HP}} = \tfrac{1}{2}[V'_{\mathrm{DM}} + (V'_{\mathrm{DM}})^\dagger] . \quad \text{(F5)}$$

Thus the term $V^{(0)}_{\mathrm{HP}}$ is identical in form to the term V_{DM} of Eq. (2.16), with the potential coefficients $\Phi^{(i)}$ replaced by functions $\tilde{\Phi}^{(i)}$, given by

$$\tilde{\Phi}^{(1)}_{1234} = \tilde{\Phi}^{(9)}_{1234} = \tfrac{1}{2}(\Phi^{(1)}_{1234} + \Phi^{(1)}_{3412}) , \quad \text{(F6a)}$$

$$\tilde{\Phi}^{(2)}_{1234} = \tilde{\Phi}^{(3)}_{3412} = \tilde{\Phi}^{(5)}_{3421} = \tilde{\Phi}^{(6)}_{2134} = \tfrac{1}{2}(\Phi^{(2)}_{1234} + \Phi^{(3)}_{3412}) , \quad \text{(F6b)}$$

$$\tilde{\Phi}^{(4)}_{1234} = \tfrac{1}{2}(\Phi^{(4)}_{1234} + \Phi^{(4)}_{3412}) , \quad \text{(F6c)}$$

$$\tilde{\Phi}^{(7)}_{1234} = \tilde{\Phi}^{(8)}_{1234} = \tfrac{1}{2}(\Phi^{(7)}_{1234} + \Phi^{(8)}_{3412}) . \quad \text{(F6d)}$$

2. Calculation of $\Sigma''_{\mu\nu}(\vec{k},\omega)$ in Born Approximation in Quantum Low-Temperature Regime

At low temperatures, we shall keep only the lowest-order term in Eq. (F4), since the other terms contain higher powers of $1/S$ and $1/z$. The matrix elements which enter the Born-approximation calculation of $\Sigma''_{\alpha\alpha}(\vec{k},\omega)$ will have the same form as in Eqs. (2.39) and (4.4) with $\Phi^{(i)}$ replaced by $\tilde{\Phi}^{(i)}$. At long wavelengths, we thus obtain the matrix elements

$$M_{22} = \tfrac{1}{8}[\epsilon_{\bar{1}}\epsilon_{\bar{2}}(\hat{k}_1\cdot\hat{k}_2 - 1) + \epsilon_{\bar{3}}\epsilon_{\bar{4}}(\hat{k}_3\cdot\hat{k}_4 - 1)]^2 + \tfrac{1}{8}[\epsilon_{\bar{1}}\epsilon_{\bar{3}}(\hat{k}_1\cdot\hat{k}_3 - 1) + \epsilon_{\bar{2}}\epsilon_{\bar{4}}(\hat{k}_2\cdot\hat{k}_4 - 1)]^2 + \tfrac{1}{8}[\epsilon_{\bar{1}}\epsilon_{\bar{4}}(\hat{k}_1\cdot\hat{k}_4 - 1) + \epsilon_{\bar{2}}\epsilon_{\bar{3}}(\hat{k}_2\cdot\hat{k}_3 - 1)]^2 , \quad \text{(F7a)}$$

$$M_{31} = \tfrac{1}{8}[\epsilon_{\bar{1}}\epsilon_{\bar{2}}(\hat{k}_1\cdot\hat{k}_2 + 1) + \epsilon_{\bar{3}}\epsilon_{\bar{4}}(\hat{k}_3\cdot\hat{k}_4 - 1)]^2 + \tfrac{1}{8}[\epsilon_{\bar{1}}\epsilon_{\bar{3}}(\hat{k}_1\cdot\hat{k}_3 + 1) + \epsilon_{\bar{2}}\epsilon_{\bar{4}}(\hat{k}_2\cdot\hat{k}_4 - 1)]^2 + \tfrac{1}{8}[\epsilon_{\bar{1}}\epsilon_{\bar{4}}(\hat{k}_1\cdot\hat{k}_4 + 1) + \epsilon_{\bar{2}}\epsilon_{\bar{3}}(\hat{k}_2\cdot\hat{k}_3 - 1)]^2 . \quad \text{(F7b)}$$

Note that in contrast to the Dyson-Maleev case, both M_{22} and M_{31} are positive. This is a consequence of the Hermitian nature of the Holstein-Primakoff Hamiltonian. We may anticipate that the cancellation which was necessary for self-consistency in Sec. IV will not occur in this case. On shell, i.e., for $\Delta\epsilon_{12}=0$, the term M_{31} does not contribute, and M_{22} agrees with the Dyson-Maleev expression, Eq. (3.18), as can be seen from Eq. (A21). It follows that whenever all four magnons are confined to the energy shell, the Born-approximation results will be identical in the two formalisms. This will be the case in regimes B–D of Sec. III for an incoming magnon which is on-resonance. Moreover, as was remarked in Sec. IV, the lowest Born approximation is self-consistent in these regimes. In regime A, the two formalisms will be identical for magnons on the energy shell, and on resonance ($\rho = 1$).[72] For arbitrary values of ρ, however, and taking into account the possibility of intermediate-state widths, we have in regime A, instead of Eq. (4.6),

$$\tfrac{1}{4} M_{22} = (\epsilon_{\vec{k}}/\epsilon_{\vec{s}})\left(\tfrac{1}{2} - \mu - \Delta/2\epsilon_{\vec{k}}\right)^2 , \tag{F8a}$$

$$\tfrac{1}{4} M_{31} = (\epsilon_{\vec{k}}/\epsilon_{\vec{s}})\left(\tfrac{1}{2} + \mu + \Delta/2\epsilon_{\vec{k}}\right)^2 , \tag{F8b}$$

and, in place of Eq. (4.20),

$$\tilde{K}(\epsilon_{\vec{k}}, \rho) = (2\pi)^{-2} \int d\vec{p} \int d\vec{s}\, n_{\vec{p}}(1 + n_{\vec{p}})\, s^{-2} \int_{-\infty}^{+\infty} (\alpha)\, d\alpha \times \left[\delta(2\epsilon_{\vec{k}}\rho + 2\Delta - \alpha)\left(\tfrac{1}{2} - \mu - \Delta/2\epsilon_{\vec{k}}\right)^2 + \delta(-2\epsilon_{\vec{k}}\rho + 2\Delta - \alpha)\left(\tfrac{1}{2} + \mu + \Delta/2\epsilon_{\vec{k}}\right)^2\right] . \tag{F9}$$

It is already clear from Eq. (F9), that for $\rho = 0$, for instance, we have a term proportional to $\epsilon_{\vec{k}}^{-2} \times \int_{-\infty}^{+\infty} \alpha^2 \varphi(\alpha)\, d\alpha = (\gamma/\epsilon_{\vec{k}})^2$, which diverges as $\epsilon_{\vec{k}} \to 0$. Thus the result of Eq. (F9) is not self-consistent for $\rho = 0$. We may evaluate Eq. (F9) for arbitrary values of ρ by the method of Appendix D. We again find an expression of the form of Eq. (D14), with F now given by

$$F(y, \mu) = \left[\mu - \tfrac{1}{2}(1 - y + \rho)\right]^2 . \tag{F10}$$

Using Eqs. (D19) and (D20) we find the result

$$\tilde{K}(\epsilon_{\vec{k}}, \rho) = I_A\left[\tfrac{1}{4} + \tfrac{3}{16}(1 + \rho)^2 + \tfrac{3}{64}(\gamma/\epsilon_{\vec{k}})^2\right] , \tag{F11}$$

where γ is defined in Eq. (4.17). This result implies that when $\gamma/\epsilon_{\vec{k}} \ll 1$:

$$\hbar^{-1} \Sigma''_{\alpha\alpha}(\vec{k}, \omega) = \tfrac{1}{8} \omega_E \epsilon_{\vec{k}}^2 \rho (2\pi)^{-3} S^{-2} I_A (7 + 6\rho + 3\rho^2) . \tag{F12}$$

Thus the Holstein-Primakoff expression for the decay rate in first Born approximation is not self-consistent in all of regime A, since there is a term in Eq. (F11) which diverges when $\epsilon_{\vec{k}} \ll \gamma$, i.e., when $E_{\vec{k}} \ll \hbar\Gamma_{\text{th}}$. According to our evaluation, however, as long as $\epsilon_{\vec{k}} \gg \gamma$, the Holstein-Primakoff answer should be correct, since then the corrections due to self-consistency are small. We note first that for $\rho = 1$, i.e., on-resonance, Eq. (F12) agrees with the Dyson-Maleev result, Eq. (4.24). Thus the decay rates are the same. On the other hand, we have asserted that the values of $\Sigma''_{\mu\nu}(\vec{k}, \omega)$ are also physically significant for $\rho \neq 1$, since they enter the correlation functions. We must therefore check that in the case $\epsilon_{\vec{k}} \gg \gamma$, the two formalisms predict the same spin-correlation functions. We shall study this question in part 6 of this Appendix.

In order to evaluate the correlation function, it will be necessary to compute $\Sigma''_{\beta\alpha}(\vec{k}, \omega)$. We shall only consider the regime $\epsilon_{\vec{k}} \gg \gamma$. As indicated in Appendix B, in order to calculate $\Sigma''_{\beta\alpha}(\vec{k}, \omega)$ we proceed as for $\Sigma''_{\alpha\alpha}(\vec{k}, \omega)$, except that in the matrix elements we make the replacements given in Eq. (B1). We find that instead of Eq. (F8) we have

$$\tfrac{1}{4} M_{22} = (\epsilon_{\vec{k}}/\epsilon_{\vec{s}})\left(\tfrac{1}{2} - \mu + \tfrac{1}{2}\rho\right)\left(\tfrac{1}{2} + \mu - \tfrac{1}{2}\rho\right) , \tag{F13a}$$

$$\tfrac{1}{4} M_{31} = (\epsilon_{\vec{k}}/\epsilon_{\vec{s}})\left(\tfrac{1}{2} + \mu + \tfrac{1}{2}\rho\right)\left(\tfrac{1}{2} - \mu - \tfrac{1}{2}\rho\right) , \tag{F13b}$$

so that now we evaluate Eq. (D14) with $F(y, \mu) \sim \frac{1}{4} - (\mu - \frac{1}{2}\rho)^2$. Then use of Eqs. (D19) and (D20) leads to the result

$$\hbar^{-1} \Sigma''_{\beta\alpha}(\vec{k}, \omega) = -\tfrac{1}{8} \omega_E \epsilon_{\vec{k}}^2 \rho (2\pi)^{-3} S^{-2} I_A (1 + 3\rho^2) . \tag{F14}$$

The other elements of the self-energy matrix can be determined using the general symmetry relations given in Eq. (2.24):

$$\Sigma''_{\alpha\beta}(\vec{k}, \omega) = -\Sigma''_{\beta\alpha}(\vec{k}, -\omega) = \Sigma''_{\beta\alpha}(\vec{k}, \omega) , \tag{F15a}$$

$$\Sigma''_{\beta\beta}(\vec{k}, \omega) = -\Sigma''_{\alpha\alpha}(\vec{k}, -\omega) . \tag{F15b}$$

3. Calculation of $\Sigma''_{\mu\nu}(\vec{k},\omega)$ in Born Approximation in Classical Low-Temperature Regime

The results in the classical regime can be inferred immediately from the above calculations, by replacing μ by $v_{\vec{p}}\mu$ [e.g., compare Eqs. (4.6a) and (6.9)]. Thus, for instance, in the classical regime and when $\epsilon_{\vec{k}} \gg \gamma$ (so that y can be neglected), we have instead of Eq. (F10)

$$F(y, \mu) \approx \left[v_{\vec{p}}\mu - \tfrac{1}{2}(1 + \rho)\right]^2 , \tag{F16}$$

which leads to the result

$$\hbar^{-1} \Sigma''_{\alpha\alpha}(\vec{k}, \omega) = \pi^{-1}(T/T_0)^2 \omega_E \epsilon_{\vec{k}}^2 \rho \left[\tfrac{1}{3}\eta_2 + \tfrac{1}{4}(1 + \rho)^2 \eta_0\right] , \tag{F17}$$

which again agrees with the Dyson-Maleev answer given in Eq. (6.19) on resonance ($\rho = 1$). For $\Sigma''_{\beta\alpha}(\vec{k}, \omega)$ we have (neglecting y, which is of order $\gamma/\epsilon_{\vec{k}}$)

$$F(y, \mu) \approx \tfrac{1}{4} - (v_{\vec{p}}\mu - \tfrac{1}{2}\rho)^2 . \tag{F18}$$

This form leads to

$$\hbar^{-1} \Sigma''_{\beta\alpha}(\vec{k}, \omega) = \pi^{-1}(T/T_0)^2 \omega_E \epsilon_{\vec{k}}^2 \rho \left[-\tfrac{1}{3}\eta_2 + \tfrac{1}{4}(1 - \rho^2)\eta_0\right] . \tag{F19}$$

Note that this result does *not* agree with the Dyson-Maleev result, Eq. (B5), even on resonance. However, agreement between the two formalisms is not required, since the off-diagonal elements of $\Sigma_{\mu\nu}$ represent higher-order corrections; the dominant diagonal terms are the same in the two formalisms, as we have already seen.

4. General Form of Spin-Correlation Functions in Holstein-Primakoff Formalism

In this section we investigate the general form of the spin-correlation functions in the Holstein-Primakoff formalism. In order to obtain the correlation functions to the desired accuracy, we need to evaluate Green's functions of the type

$$\mathcal{G}_{aa}(\vec{k}, z) \equiv \langle\langle a_{\vec{k}} + \tfrac{1}{2}A_{\vec{k}};\ a^{\dagger}_{\vec{k}} + \tfrac{1}{2}A^{\dagger}_{\vec{k}}\rangle\rangle, \quad \text{etc.} \quad , \qquad \text{(F20)}$$

where $A_{\vec{k}}$ is defined in Eq. (2.29). By diagrammatic reasoning analogous to that of Appendix C we see that we can write

$$\underline{\mathcal{G}} = (\underline{1} + \tfrac{1}{2}\underline{\Lambda}^{\rm out})\ \underline{G}\ (\underline{1} + \tfrac{1}{2}\underline{\Lambda}^{\rm in}) + \tfrac{1}{4}\underline{W} \quad , \qquad \text{(F21)}$$

where $\underline{\Lambda}^{\rm out}(\vec{k}, z)$ and $\underline{\Lambda}^{\rm in}(\vec{k}, z)$ are self-energy matrices in which the outgoing and incoming vertices are replaced by $V_{\rm eff}$ and $V^{\dagger}_{\rm eff}$, respectively. Also, $\underline{W}$ is a self-energy in which both external vertices are replaced by these effective interactions. Diagrammatically, this equation is shown in Fig. 9. As we have seen above, the Holstein-Primakoff and Dyson-Maleev results agree on resonance, because they give identical results for $\Sigma''_{\alpha\alpha}(\vec{k}, \epsilon_{\vec{k}})$, which determines the resonant behavior of $\mathcal{G}_{\alpha\alpha}(\vec{k}, \omega)$ for $\omega \approx \omega_E\,\epsilon_{\vec{k}}$. On resonance, the other functions are of higher order, i.e.,

$$\operatorname{Im}\mathcal{G}_{\mu\nu}(\vec{k}, \epsilon_{\vec{k}}) / \operatorname{Im}\mathcal{G}_{\alpha\alpha}(\vec{k}, \epsilon_{\vec{k}}) \sim \mathcal{O}(\Sigma) \approx k^2\tau^3 \quad , \qquad \text{(F22)}$$

unless $\mu = \nu = \alpha$. The fact that the two formalisms give different results for $\Sigma''_{\mu\nu}(\vec{k}, \epsilon_{\vec{k}})$ except when $\mu = \nu = \alpha$, is a higher-order effect about which our calculation can say nothing. Hence in this discussion we need only concern ourselves with the nonresonant behavior of the correlation functions.

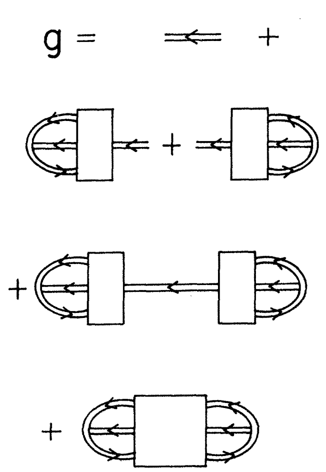

FIG. 9. Expansion of the spin Green's function in the Holstein-Primakoff formalism. Here we show the diagrammatic representation of Eq. (F21), which takes account of terms up to third order in the Holstein-Primakoff expansion of the spin operators in terms of boson operators. The double lines represent true boson propagators. The vertex functions are assumed to be irreducible (i.e., they cannot be partitioned by cutting only a single line), so that the reducible contribution to the spin Green's function in the next-to-lowest line must be given explicitly.

For this purpose, we shall discuss the properties of $\Lambda^{\rm out}_{\mu\nu}(\vec{k}, z)$ and $\Lambda^{\rm in}_{\mu\nu}(\vec{k}, z)$. We wish to express $\Lambda^{\rm in}_{\mu\nu}(\vec{k}, z)$ in terms of $\Lambda^{\rm out}_{\mu\nu}(\vec{k}, z)$, since the latter quantity has been discussed at length in Appendix C. First of all, it is clear from their diagrammatic structure that

$$\Lambda^{\rm out}_{\alpha\alpha}(\vec{k}, z) = \Lambda^{\rm in}_{\alpha\alpha}(\vec{k}, z) \quad . \qquad \text{(F23)}$$

Second, we note that

$$[\underline{\Lambda}^{\rm out}(\vec{k}, z)\ \underline{G}\ (\vec{k}, z)]_{ab} = \langle\langle A_{\vec{k}}; b_{-\vec{k}}\rangle\rangle \quad , \qquad \text{(F24)}$$

$$[\underline{G}(\vec{k}, z)\ \underline{\Lambda}^{\rm in}(\vec{k}, z)]_{ab} = \langle\langle a_{\vec{k}}; B_{-\vec{k}}\rangle\rangle \quad . \qquad \text{(F25)}$$

But using the symmetry in the Holstein-Primakoff Hamiltonian between $a_{\vec{k}}$ and $b_{-\vec{k}}$, we have that

$$\langle\langle a_{\vec{k}}; B_{-\vec{k}}\rangle\rangle = \langle\langle b_{-\vec{k}}; A_{\vec{k}}\rangle\rangle \quad , \qquad \text{(F26)}$$

so that

$$[\underline{G}(\vec{k}, z)\,\underline{\Lambda}^{\rm in}(\vec{k}, z)]_{ab} = [\underline{\Lambda}^{\rm out}(\vec{k}, -z)\,\underline{G}(\vec{k}, -z)]_{ab} \quad . \qquad \text{(F27)}$$

In a similar fashion, one can deduce that

$$[\underline{G}(\vec{k}, z)\ \Lambda^{\rm in}(\vec{k}, z)]_{\alpha\beta} = [\underline{\Lambda}^{\rm out}(\vec{k}, -z)\ \underline{G}(\vec{k}, -z)]_{\alpha\beta} \quad . \qquad \text{(F28)}$$

Thus, finally,

$$\tfrac{1}{2}\operatorname{Im}[\underline{\Lambda}^{\rm out}(\vec{k}, \omega)\ \underline{G}(\vec{k}, \omega) + \underline{G}(\vec{k}, \omega)\ \underline{\Lambda}^{\rm in}(\vec{k}, \omega)]_{\alpha\beta}$$
$$= \tfrac{1}{2}\operatorname{Im}[\underline{\Lambda}^{\rm out}(\vec{k}, \omega)\,\underline{G}(\vec{k}, \omega) - \underline{\Lambda}^{\rm out}(\vec{k}, -\omega)\,\underline{G}(\vec{k}, -\omega)]_{\alpha\beta} \quad . \qquad \text{(F29)}$$

Using these results and carrying out a calculation analogous to that of Sec. VII, we find the correlation functions off resonance to be given to lowest order in τ and $(zS)^{-1}$ as

$$\operatorname{Im}\mathcal{G}_{\alpha\alpha}(\vec{k}, \omega) = \tfrac{1}{4}W''_{\alpha\alpha}(\vec{k}, \omega) + H_E^{-2}\,\Sigma''_{\alpha\alpha}(\vec{k}, \omega)\,(\tilde{\omega} - \epsilon_{\vec{k}})^{-2} + H_E^{-1}\,\Lambda''_{\alpha\alpha}(\vec{k}, \omega)\,(\tilde{\omega} - \epsilon_{\vec{k}})^{-1} \quad , \qquad \text{(F30a)}$$

$$\operatorname{Im}\mathcal{G}_{\beta\alpha}(\vec{k}, \omega) = \tfrac{1}{4}W''_{\beta\alpha}(\vec{k}, \omega) - H_E^{-2}\,\Sigma''_{\beta\alpha}(\vec{k}, \omega)\,(\tilde{\omega}^2 - \epsilon^2_{\vec{k}})^{-1} + \tfrac{1}{2}H_E^{-1}\,\Lambda''_{\beta\alpha}(\vec{k}, \omega)\,(\tilde{\omega} - \epsilon_{\vec{k}})^{-1} - \tfrac{1}{2}H_E^{-1}\,\Lambda''_{\alpha\beta}(\vec{k}, \omega)\,(\tilde{\omega} + \epsilon_{\vec{k}})^{-1} \quad . \qquad \text{(F30b)}$$

Here we used the notation $\Lambda_{\mu\nu}(\vec{k}, z) \equiv \Lambda^{\rm out}_{\mu\nu}(\vec{k}, z)$. Thus $\Lambda_{\mu\nu}(\vec{k}, z)$ is calculated as in Appendix C, except that the matrix elements are constructed using the interaction appropriate to the Holstein-Primakoff Hamiltonian, Eqs. (F6).

5. Calculation of $\Lambda''_{\mu\nu}(\vec{k}, \omega)$ and $W_{\mu\nu}(\vec{k}, \omega)$

We need to calculate the vertex function $\Lambda_{\mu\nu}(\vec{k}, z)$ and the self-energy $W_{\mu\nu}(\vec{k}, z)$. Let us first consider $\Lambda''_{\mu\nu}(\vec{k}, z)$. We may modify the calculation of Ap-

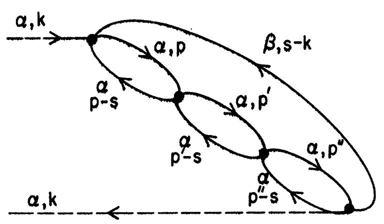

FIG. 13. Anomalous contribution to the decay rate. The singular behavior of this diagram is due to the particle-hole ladder discussed in the text [see Eq. (G27)].

(4.20), one finds

$$K(\epsilon_k, \rho) \sim \int \varphi(\alpha)d\alpha \int \varphi(\alpha')d\alpha' \int \varphi(\alpha'')d\alpha''$$
$$\times \int d\vec{p} \int d\vec{p}' \int d\vec{p}'' \int ds(\tau/\epsilon_{\vec{s}})(M/\epsilon_{\vec{k}})$$
$$\times n_{\vec{p}}(1+n_{\vec{p}})n_{\vec{p}'}(1+n_{p'})n_{\vec{p}''}(1+n_{\vec{p}''})$$
$$\times \delta(2\Delta' - \alpha')\delta(2\Delta'' - \alpha'') . \qquad (G27)$$

In writing this result we have used the fact that p, p', and p'' are thermal momenta much larger than s. Also, M denotes the matrix element for this process which we estimate to be of order $\gamma/\epsilon_{\vec{s}}$, and the Δ's in the energy δ functions are given by analogs of Eq. (4.22). From that equation we see that the integration over the orientations of the momenta leads to integrals over ν, ν', and ν'', each of which produce factors of $1/s$. Thus the s integral in Eq. (G27), denoted I_s, is highly singular:

$$I_s = \int_{s_1}^{s_2} s^{-3}\, ds . \qquad (G28)$$

Compared to Eq. (D5a), we have a factor s^{-3}, rather than s^{-1}, because there are three energy denominators in the particle-hole ladder instead of one as in the first Born approximation. Clearly, with more rungs in the ladder we could generate even more singular factors. Since $s_1 \sim \gamma$, this integral gives the anomalously large contribution

$$I_s \sim \gamma^{-2} \sim \tau^{-10}\,! \qquad (G29)$$

As mentioned in Sec. V, to avoid this difficulty it is clear that one must perform a resummation over particle-hole ladders, replacing them by a finite scattering amplitude. It is just such a sum over particle-hole ladders which describes the longitudinal susceptibilty, so it is not surprising that the perturbation expansion of this four-point function does not converge. Furthermore, from this example, we may conclude that for a complete analysis it would be necessary to consider the possibility of resonances in many-point functions. Clearly, the analysis of such general collective excitations is beyond the reach of our formalism.

3. Absence of Anomalous Temperature Renormalizations

Let us now accept the validity of the $\epsilon_{\vec{k}}$ expansion, and write the decay rate in the form

$$\Gamma(\vec{k}, \omega) = \omega_E S^{-2}\epsilon_{\vec{k}}^2 g(\rho, \tau), \quad \epsilon_{\vec{k}} \ll 1 . \qquad (G30)$$

The assumption of the validity of this result simplifies the analysis in that now we can restrict our attention to terms of lowest order in τ. Indeed, terms of higher order in τ are only dangerous if they lead to a dependence on $\epsilon_{\vec{k}}$ of lower order than $\epsilon_{\vec{k}}^2$, but such a dependence has been excluded in writing Eq. (G30). For the present discussion, we shall assume that contributions to the decay rate from interactions with collective modes, such as second magnons, can be ignored. Mathematically, this assumption means that perturbation theory gives qualitatively correct results and that terms which are of higher order in some expansion parameter can be neglected.

Taking this point of view, one is led to the conclusion that the term of lowest order in the density (of spin deviations) is found by considering the family of diagrams with the minimum number of hole lines. In other words, to lowest order in τ we need consider only those diagrams which involve no hole lines other than those required for energy conservation. Furthermore, processes with more than one energy-conserving δ function involve extra thermal magnons, and hence give contributions which are of higher order in τ. In addition, it is clear that to achieve the minimum number of hole lines, the energy-conserving δ function should involve four particles in just the same way as the lowest Born approximation. Energy conservation for six particles, for example, requires two extra independent momenta to be thermal, so that effectively there are two extra hole lines in this case. From these remarks we conclude that to lowest order in τ, only diagrams similar to those in Fig. 4 contribute to the decay rate. In other words, to lowest order in τ we replace the vertices $\Phi^{(i)}$ by vertices $R^{(i)}$ which are obtained by "dressing" the $\Phi^{(i)}$ with subdiagrams involving no extra hole lines. Note that all energy denominators other than the δ function "cut," which we have explicitly represented in Fig. 4 by a dashed line, involve some nonthermal particle lines. Since the momenta of these internal particle lines range over the entire Brillouin zone, the inclusion of damping for such intermediate states leads to negligible corrections at low temperatures. In short, we can evaluate the scattering amplitudes or dressed vertices at zero temperature. As we shall see later in this Appendix, and also more explicitly in Appendix H, these dressed vertices are given as a series in the parameters z^{-1} and $(zS)^{-1}$.

As mentioned in Sec. V B, we shall assume that these vertex functions satisfy "Hermiticity" relations like Eq. (5.10). These equalities are strongly documented by evidence presented in Sec. V C and Appendix H. Therefore, the estimates of Eq. (5.5) and (5.6) will still be valid for the renormalized vertices. In particular, in Eq. (5.9) the matrix elements $\mathfrak{M}_{22}$ and $\mathfrak{M}_{31}$ will be of the form

$$\mathfrak{M} \sim \epsilon_{\vec{s}}^{-1}(A\epsilon_{\vec{k}} + B\tilde{\omega} + C\alpha). \qquad \text{(G31)}$$

In writing down this estimate we have dropped terms of higher order in $\epsilon_{\vec{k}}/\epsilon_{\vec{s}}$ or $\alpha/\epsilon_{\vec{s}}$. Using this form of matrix element, one can carry out the integrals over the scattering surface in the manner of Appendix D, and show that the term $C\nu'$, involving the damping of intermediate states does not contribute to the result. In other words, the matrix elements can be evaluated on shell, in which case we may estimate them as follows: The dressed interaction R_{in} must be linear in k and p, and hence is of the form

$$R_{\text{in}} \sim (A' + B'\mu)kp\ , \qquad \text{(G32a)}$$

where $\mu = \hat{k}\cdot\hat{p}$. On resonance ($\rho = 1$) we assume that $R_{\text{out}} = R_{\text{in}}$, so that in general we have

$$R_{\text{out}} \sim [A' + B'\mu + C'(\rho - 1)]\ kp, \qquad \text{(G32b)}$$

which enables us to write

$$\mathfrak{M}(\vec{k}, \vec{p}, \vec{r}, \vec{s}) \sim (\epsilon_{\vec{k}}/\epsilon_{\vec{s}})(\lambda' + \beta'_\mu)[\lambda' + \gamma'_\mu + C'(\rho - 1)]. \qquad \text{(G32c)}$$

But this matrix element leads to exactly the same temperature dependence, viz., $\Sigma''_{\alpha\alpha}(\vec{k}, \omega) \sim \tau^3 \ln \tau$, as was found in lowest Born approximation [see Eq. (5.1)]. Thus we argue that higher-order terms will only give rise to $1/z$ corrections to $f(\rho)$ in Eq. (5.1), and hence that the first Born approximation is qualitatively correct.

In order to give this discussion in more detail, it is necessary to clarify some points connected with the enumeration of diagrams. For this purpose we shall study the symmetry number,[111] $g(D)$, of the diagram D. The symmetry number is the number of ways of relabeling the lines which leaves the diagram topologically unchanged. In particular, consider the diagrms for $R^{(1)}$ shown in Fig. 14. If we denote by m, n the full diagram for Σ'', obtained by using the diagram m for $R^{(1)}$ at the incoming vertex, and n for $R^{(1)}$ at the outgoing vertex, then we have

$$g(m, 4) = 2g(m)g(4), \quad m = 1, 3 \qquad \text{(G33a)}$$

but for any other combinations of the diagrams of Fig. 14,

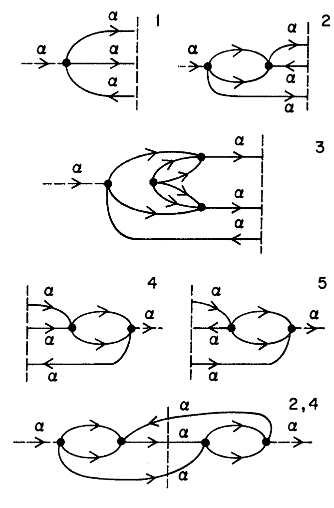

FIG. 14. Some dressed vertices for the self-energy. Any self-energy diagram with a single δ-function cut, such as that shown at the bottom, can be constructed from dressed vertices. In the first five diagrams we show dressed vertices involving four α magnons. All lines internal to the vertex are particle lines. Diagrams 1, 3, and 4 are "symmetric" in the sense of Appendix G3, whereas 2 and 5 are "unsymmetric".

$$g(m, n) = g(m)g(n)\ . \qquad \text{(G33b)}$$

Thus we must separate the vertex functions into contributions from diagrams which have two equivalent external lines, which we shall call "symmetric," and those from diagrams which do not have two equivalent external lines, which we shall call "unsymmetric." The symmetric and unsymmetric parts of $R^{(1)}$ will be denoted $S^{(1)}$ and $U^{(1)}$, respectively.[112] Taking account of the symmetry number, we write the matrix elements in Eq. (5.9) as

$$\begin{aligned}\mathfrak{M}_{22}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = 8l_{\vec{1}}^2 l_{\vec{2}}^2 l_{\vec{3}}^2 l_{\vec{4}}^2 [&2R^{(1)}_{\vec{3}\vec{4}\vec{1}\vec{2}}(\omega)R^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}}(\omega) \\ &- S^{(1)}_{\vec{3}\vec{4}\vec{1}\vec{2}}(\omega)S^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}}(\omega) + R^{(4)}_{\vec{3},-\vec{2},\vec{1},-\vec{4}}(\omega)R^{(4)}_{\vec{1},-\vec{4},\vec{3},-\vec{2}}(\omega) \\ &+ R^{(4)}_{\vec{4},-\vec{2},\vec{1},-\vec{3}}(\omega)R^{(4)}_{\vec{1},-\vec{3},\vec{4},-\vec{2}}(\omega)]\end{aligned} \qquad \text{(G34a)}$$

and

$$\begin{aligned}\mathfrak{M}_{31}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = 8l_{\vec{1}}^2 l_{\vec{2}}^2 l_{\vec{3}}^2 l_{\vec{4}}^2 [&2R^{(5)}_{\vec{4}\vec{3}\vec{1}\vec{2}}(\omega)R^{(6)}_{\vec{1}\vec{2}\vec{3}\vec{4}}(\omega) \\ &- S^{(5)}_{\vec{4}\vec{3}\vec{1}\vec{2}}(\omega)S^{(6)}_{\vec{1}\vec{2}\vec{3}\vec{4}}(\omega) + R^{(2)}_{\vec{2},-\vec{3},-\vec{1},\vec{4}}(\omega)R^{(3)}_{-\vec{1},\vec{4},\vec{2},-\vec{3}}(\omega) \\ &+ R^{(2)}_{\vec{2},-\vec{4},-\vec{1},\vec{3}}(\omega)R^{(3)}_{-\vec{1},\vec{3},\vec{2}-\vec{4}}(\omega)]\ .\end{aligned} \qquad \text{(G34b)}$$

The frequency dependence of the dressed vertices arises because the energy associated with the external line (always $\vec{k}_1$) is $\hbar\omega$.

To illustrate the $1/z$ expansion for the renormalized vertices we have evaluated the dressed vertices on shell to first order in $1/z$, using diagrams typified by those of Fig. 15. We have obtained the following results:

$$S^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}}=[1-(2zS)^{-1}]\Phi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \qquad \text{(G35a)}$$

$$U^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}}=(2zS)^{-1}\Phi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \;, \qquad \text{(G35b)}$$

$$R^{(4)}_{\vec{1}\vec{2}\vec{3}\vec{4}}=[1-(2zS)^{-1}]\Phi^{(4)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \;, \qquad \text{(G35c)}$$

$$S^{(5)}_{\vec{1}\vec{2}\vec{3}\vec{4}}=[1-(2zS)^{-1}]\Phi^{(5)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \;, \qquad \text{(G35d)}$$

$$U^{(5)}_{\vec{1}\vec{2}\vec{3}\vec{4}}=(2zS)^{-1}\Phi^{(5)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \;, \qquad \text{(G35e)}$$

$$R^{(2)}_{\vec{1}\vec{2}\vec{3}\vec{4}}=[1+(2zS)^{-1}]\Phi^{(2)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \;, \qquad \text{(G35f)}$$

$$R^{(3)}_{\vec{1}\vec{2}\vec{3}\vec{4}}=[1+(2zS)^{-1}]\Phi^{(3)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \;. \qquad \text{(G35g)}$$

Since these dressed vertices have the same long-wavelength behavior as the bare vertices, Eqs. (G31) and (G32) are obviously satisfied, to this order in $1/z$.

4. Other Functions

A completely analogous discussion can be given for the real part of the self-energy $\Sigma'_{\alpha\alpha}(\vec{k},\omega)$. Instead of Eq. (G7) one finds

$$\Sigma'_{\alpha\alpha}(\vec{k},\omega)=\tfrac{1}{2}\textstyle\sum_D[\Sigma'_{\alpha\alpha}(D;\vec{k},\omega)+\Sigma'_{\alpha\alpha}(D';\vec{k},\omega)] \;. \qquad \text{(G36)}$$

In this case, note that the detailed-balance factor $(1-e^{-\hbar\beta\omega})\sim\hbar\beta\omega$ is missing. Also, one has an even number of δ functions, so that

$$\Delta'(\vec{k},\omega)=\Delta(-\vec{k},-\omega) \;. \qquad \text{(G37)}$$

The differences in signs between Eqs. (G7) and (G13) on the one hand and (G36) and (G37) on the other hand, compensate one another. Thus the previous discussion can be used to support the claim

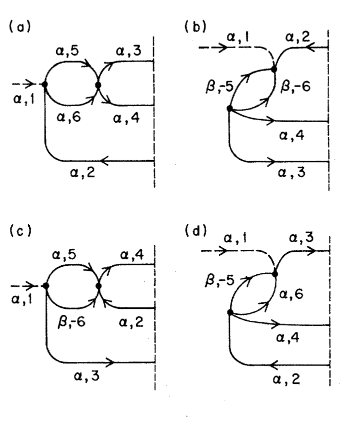

FIG. 15. First-order dressed vertices for $S^{(1)}$ [(a) and (b)] and $U^{(1)}$ [(c) and (d)] at zero temperature. All lines internal to the vertex are particle lines. The rules for evaluating the dressed vertices follow directly from those for the self-energy.

that the square bracket in Eq. (G36) is of order $\epsilon_{\vec{k}}$.

We may discuss the temperature dependence of $\Sigma'_{\alpha\alpha}(\vec{k},\omega)$ in close analogy with the discussion in part 3 of this Appendix. Since we have established that $\Sigma'_{\alpha\alpha}(\vec{k},\omega)\sim\epsilon_{\vec{k}}$, it suffices to restrict our attention to contributions to $\Sigma'_{\alpha\alpha}(\vec{k},\omega)$ which are of lowest order in τ. That is, we need consider only diagrams with no hole lines, and consequently no energy-conserving δ functions. Thus, in the expression for $\Sigma'_{\alpha\alpha}(\vec{k},\omega)$ similar to Eq. (G8), there does not appear the factor $\Delta(\{\vec{k}_i\},\{\omega_i\},\vec{k},\omega)$. As a result, it is clear that the external momentum and energy, $\vec{k}$ and ω, appear only in combinations of the form

$$\omega+\omega_E\epsilon_{\vec{k}}+\textstyle\sum_i\omega_i \;, \qquad \text{(G38a)}$$

$$\vec{k}+\textstyle\sum_i\vec{k}_i \;, \qquad \text{(G38b)}$$

as can be seen from analysis of the expressions for the other factors, P, L, N, Φ, etc., in Eq. (G8). Note that there are no δ functions to restrict the summations, which therefore range over the entire Brillouin zone. Accordingly, $\tilde{\omega}_i$ and $\vec{k}_i$ are of order unity. As a result, we may write

$$\Sigma'_{\alpha\alpha}(D;\vec{k},\omega)=\sum_{nm}A_{nm}\omega^n\epsilon^m_{\vec{k}} \;, \qquad \text{(G39)}$$

where the coefficients A_{nm} are of order unity. Likewise, we may write

$$\Sigma'_{\alpha\alpha}(D';\vec{k},\omega)=\sum_{nm}A'_{nm}\omega^n\epsilon^m_{\vec{k}} \;, \qquad \text{(G39b)}$$

and the cancellation theorem ensures that $A_{00}=-A'_{00}$. Thus we have established that

$$\Sigma'_{\alpha\alpha}(\vec{k},\omega)=\hbar\,\omega_E\epsilon_{\vec{k}}(C_3+C_4\rho) \;, \qquad \text{(G40)}$$

in the zero-temperature long-wavelength limit, where C_3 and C_4 are constants.

Up to now we have considered explicitly only $\Sigma_{\alpha\alpha}(\vec{k},z)$. But it is easy to see that our arguments go through unchanged for any of the components, $\Sigma_{\mu\nu}(\vec{k},z)$.

We can carry out a similar type of analysis for $\underline{\Lambda}(\vec{k},z)$, except that in this case the arguments simplify, because there are no cancellations analogous to those in $\underline{\Sigma}(\vec{k},\omega)$. As explained in Appendix C, we use the matrix element $\hat{\Phi}$ at the outgoing vertex, so that Φ_{out} is still of order unity, cf. Eqs. (G21) and (C12). Since there is no cancellation here, one sees that $\epsilon_{\vec{k}}\underline{\Lambda}$ and $\underline{\Sigma}$ are of the same order:

$$\underline{\Lambda}'(\vec{k},\omega)\sim\mathcal{O}(1) \;, \qquad \text{(G41a)}$$

$$\underline{\Lambda}''(\vec{k},\omega)\sim\epsilon_{\vec{k}} \;. \qquad \text{(G41b)}$$

We are now in a position to develop a formally exact description of the self-energies and vertex functions at long wavelengths and low temperatures, without any restriction on the parameters z and S. We write

$$\Sigma''_{\mu\nu}(\vec{k},\omega)=\epsilon_{\vec{k}}^2 f^{(2)}_{\mu\nu}(\rho,\tau)\ , \qquad \text{(G42a)}$$

$$\Lambda''_{\mu\nu}(\vec{k},\omega)=\epsilon_{\vec{k}} h^{(2)}_{\mu\nu}(\rho,\tau)\ , \qquad \text{(G42b)}$$

$$\Sigma'_{\mu\nu}(\vec{k},\omega)=\epsilon_{\vec{k}} f^{(1)}_{\mu\nu}(\rho,\tau)\ , \qquad \text{(G42c)}$$

$$\Lambda'_{\mu\nu}(\vec{k},\omega)=h^{(1)}_{\mu\nu}(\rho,\tau), \qquad \text{(G42d)}$$

and we wish to determine the leading temperature dependence of these functions. To do this we need consider only diagrams with the minimum number of hole lines. From the discussion following Eq. (G32), it is clear that we may write

$$\Sigma''_{\alpha\alpha}(\vec{k},\omega)=\hbar\,\omega_E\epsilon_{\vec{k}}^2(C_1\rho+C_2\rho^2)\ , \qquad \text{(G43)}$$

where C_1 and C_2 are parameters proportional to $\tau^3\ln\tau$ and independent of $\vec{k}$ and ω which reduce to C_1^0 and C_2^0 in lowest order in $1/z$ and $1/S$. Using the general symmetry relations we also have that

$$\Sigma''_{\beta\beta}(\vec{k},\omega)=-\Sigma''_{\alpha\alpha}(\vec{k},-\omega)=\hbar\,\omega_E\epsilon_{\vec{k}}^2(C_1\rho-C_2\rho^2). \qquad \text{(G44)}$$

From Appendix B, we see that in lowest Born approximation $\Sigma''_{\beta\alpha}(\vec{k},\omega)=-\Sigma''_{\alpha\alpha}(\vec{k},\omega)$. Since this equality was a result of $\Phi_{\text{out}}=-\Phi'_{\text{out}}$ [see Eq. (B2) or (G23)], it will be valid to arbitrary order in perturbation theory. Hence we have

$$\Sigma''_{\beta\alpha}(\vec{k},\omega)=-\hbar\,\omega_E\epsilon_{\vec{k}}^2(C_1\rho+C_2\rho^2)\ , \qquad \text{(G45a)}$$

$$\Sigma''_{\alpha\beta}(\vec{k},\omega)=-\hbar\,\omega_E\epsilon_{\vec{k}}^2(C_1\rho-C_2\rho^2)\ . \qquad \text{(G45b)}$$

The analysis for $\Lambda''_{\mu\nu}$ is essentially the same, so we quote only the results:

$$\Lambda''_{\mu\nu}(\vec{k},\omega)=C_5\bar{\omega},\quad \mu=\alpha,\beta,\quad \nu=\alpha,\beta \qquad \text{(G46)}$$

where C_5 is again a parameter proportional to $\tau^3\ln\tau$ and independent of $\vec{k}$ and ω.

Next we consider the real parts of these functions. From Eq. (G40), we have that

$$\Sigma'_{\alpha\alpha}(\vec{k},\omega)=\hbar\,\omega_E\epsilon_{\vec{k}}(C_3+C_4\rho)\ , \qquad \text{(G47)}$$

where C_3 and C_4 are constants independent of $\vec{k}$, ω, and τ. Use of the general symmetry relations yields,

$$\Sigma'_{\beta\beta}(\vec{k},\omega)=\Sigma'_{\alpha\alpha}(\vec{k},-\omega)=\hbar\,\omega_E\epsilon_{\vec{k}}(C_3-C_4\rho)\ . \qquad \text{(G48)}$$

To determine $\Sigma'_{\alpha\beta}$ and $\Sigma'_{\beta\alpha}$ we use the following dispersion relations:

$$\underline{\Sigma}'(\vec{k},\omega)-\underline{\Sigma}'(\vec{k},\infty)=\frac{1}{\pi}\int_{-\infty}^{+\infty}P\frac{d\omega'}{\omega-\omega'}\underline{\Sigma}''(\vec{k},\omega')\ . \qquad \text{(G49)}$$

But at long wavelengths, we have[113] $\Sigma''_{\beta\alpha}(\vec{k},\omega)=-\Sigma''_{\alpha\alpha}(\vec{k},\omega)$, hence it follows from Eq. (G49) that $\Sigma'_{\beta\alpha}(\vec{k},\omega)=-\Sigma'_{\alpha\alpha}(\vec{k},\omega)$. Thus we have

$$\Sigma'_{\beta\alpha}(\vec{k},\omega)=-\hbar\,\omega_E\epsilon_{\vec{k}}(C_3+C_4\rho)\ , \qquad \text{(G50a)}$$

$$\Sigma'_{\alpha\beta}(\vec{k},\omega)=-\hbar\,\omega_E\epsilon_{\vec{k}}(C_3-C_4\rho)\ . \qquad \text{(G50b)}$$

For Λ', the analysis is similar to that for Σ', expect that one should write (for notation, see Appendix C)

$$2S\mathcal{G}_{ab}-2SG_{ab}=-N^{-1}\sum_{\vec{p}\vec{r}}\langle\langle a^\dagger_{\vec{p}+\vec{r}}\,a_{\vec{r}}\,a_{\vec{k}+\vec{p}}\,;\,b_{-\vec{k}}\rangle\rangle\ , \qquad \text{(G51a)}$$

$$2S\mathcal{G}_{ab}-2SG_{ab}=-2N^{-1}\sum_{\vec{r}}\langle a^\dagger_{\vec{r}}\,a_{\vec{r}}\rangle\,G_{ab}-N^{-1}\sum_{\vec{p}\vec{r}}\langle\langle a^\dagger_{\vec{p}+\vec{r}}\,a_{\vec{r}}\,a_{\vec{k}+\vec{r}}\,;\,b_{-\vec{k}}\rangle\rangle_{\text{con}}\ , \qquad \text{(G51b)}$$

where the subscript "con" denotes contributions from connected diagrams. In terms of the relative sublattice magnetization σ, let us write

$$\mathcal{G}_{ab}(\vec{k},\omega)=\sigma G_{ab}(\vec{k},\omega)+\sum_m\lambda_{am}(\vec{k},\omega)G_{mb}(\vec{k},\omega)\ . \qquad \text{(G52)}$$

From Eq. (G51), we see that the diagrams for $\lambda(\vec{k},\omega)$ are connected. The analysis of these diagrams is then straightforward, and we find that

$$\delta_{\mu\nu}+\Lambda'_{\mu\nu}=\sigma\delta_{\mu\nu}+C_6\ , \qquad \text{(G53)}$$

where $\delta_{\mu\nu}$ is the Kronecker δ and σ and C_6 are constants, independent of $\vec{k}$, ω, and τ.

Thus we have achieved a complete parametrization of the dynamics in the long-wavelength limit at low temperatures in terms of the parameters $C_1,\ldots,C_6$, and σ:

$$\begin{aligned}\Sigma_{\alpha\alpha}(\vec{k},\omega-iO^+)&=-\Sigma_{\beta\alpha}(\vec{k},\omega-iO^+)\\&=-\Sigma_{\alpha\beta}(\vec{k},-\omega+iO^+)\\&=\Sigma_{\beta\beta}(\vec{k},-\omega+iO^+)=\hbar\omega_E\epsilon_{\vec{k}}(C_3+C_4\rho)\\&\quad+i\hbar\omega_E\epsilon_{\vec{k}}^2(C_1\rho+C_2\rho^2)\ ,\end{aligned} \qquad \text{(G54)}$$

$$\begin{aligned}1+\Lambda_{\alpha\alpha}(\vec{k},\omega-iO^+)&=1+\Lambda_{\beta\beta}(\vec{k},-\omega+iO^+)\\&=\sigma+C_6+iC_5\bar{\omega}\ ,\end{aligned} \qquad \text{(G55a)}$$

$$\begin{aligned}\Lambda_{\alpha\beta}(\vec{k},\omega-iO^+)&=\Lambda_{\beta\alpha}(\vec{k},-\omega+iO^+)\\&=C_6+iC_5\bar{\omega}\ .\end{aligned} \qquad \text{(G55b)}$$

To actually evaluate these parameters in terms of the exchange constant J would involve summing an infinite series in the parameters $1/z$ or $1/S$. In principle, it is necessary to determine C_3 and C_4 self-consistently. Here self-consistency does not involve treating the damping of intermediate states, as these effects are negligible at low temperatures. Rather, we must renormalize the undamped spin-wave energies by the $1/z$ effects. After this has been done, the other parameters can be determined in terms of these dressed spin-wave energies. Except for a few low-order calculations to illustrate some features of this expansion [see Appendixes H and I and Eq. (G35)], we shall not attempt a more quantitative analysis.

APPENDIX H: HERMITICITY OF DRESSED VERTICES ON RESONANCE: ANTIFERROMAGNETS AND FERROMAGNETS

1. Antiferromagnet with First-Order Dressed Vertices

In this section, we shall verify the conjectured Hermiticity property [Eq. (5.10)] for some first-order dressed vertices on shell. Since we consider only terms of leading order in k_BT/JS, we restrict the discussion to the dressed vertices $R^{(i)}$ with no hole lines, which were introduced in Sec. V B and Appendix G 3.

We have not been able to develop a general procedure for this discussion, and hence we shall merely show that a subset of diagrams for one of the dressed vertices does have the Hermitian form on resonance. We shall consider corrections only of first order in $1/S$, but we shall treat the resulting expressions exactly with regard to their dependence on $1/z$, in contrast to the results of Eq. (G35) where only leading terms in both $1/z$ and $1/S$ were kept. By carrying out the calculation of the dressed vertex to first order in $1/S$, we are effectively calculating terms in third-order perturbation theory.

We shall examine those diagrams which correspond to scattering processes involving only α magnons. To first order in $1/S$, the relevant diagrams are those of Fig. 15, from which we obtain[114] for the symmetric and unsymmetric dressed vertices, respectively,

$$S_{\vec{3}\vec{4}\vec{1}\vec{2}}(\omega)=\Phi^{(1)}_{\vec{3}\vec{4}\vec{1}\vec{2}}-(2NS)^{-1}\sum_{\vec{5}\vec{6}}\delta_{\vec{k}}(\vec{k}_5+\vec{k}_6-\vec{k}_1-\vec{k}_2)\times l_{\vec{5}}^2 l_{\vec{6}}^2\left(\frac{\Phi^{(1)}_{\vec{3}\vec{4}\vec{5}\vec{6}}\Phi^{(1)}_{\vec{5}\vec{6}\vec{1}\vec{2}}}{\tilde{\omega}+\epsilon_{\vec{2}}-\epsilon_{\vec{5}}-\epsilon_{\vec{6}}}-\frac{\Phi^{(7)}_{\vec{3}\vec{4}\vec{5}\vec{6}}\Phi^{(8)}_{\vec{5}\vec{6}\vec{1}\vec{2}}}{\epsilon_{\vec{3}}+\epsilon_{\vec{4}}+\epsilon_{\vec{5}}+\epsilon_{\vec{6}}}\right) \tag{H1}$$

and

$$U_{\vec{3}\vec{4}\vec{1}\vec{2}}(\omega)=\tfrac{1}{2}(V_{\vec{3}\vec{4}\vec{1}\vec{2}}+V_{\vec{4}\vec{3}\vec{1}\vec{2}}) , \tag{H2a}$$

where

$$V_{\vec{3}\vec{4}\vec{1}\vec{2}}=-(NS)^{-1}\sum_{\vec{5}\vec{6}}\delta_{\vec{k}}(\vec{k}_3+\vec{k}_5-\vec{k}_1-\vec{k}_6)\,l_{\vec{5}}^2 l_{\vec{6}}^2\times\left(\frac{\Phi^{(3)}_{\vec{3}\vec{5}\vec{1}\vec{6}}\Phi^{(2)}_{\vec{4}\vec{6}\vec{2}\vec{5}}}{\tilde{\omega}-\epsilon_{\vec{3}}-\epsilon_{\vec{5}}-\epsilon_{\vec{6}}}+\frac{\Phi^{(2)}_{\vec{3}\vec{5}\vec{1}\vec{6}}\Phi^{(3)}_{\vec{4}\vec{6}\vec{2}\vec{5}}}{\epsilon_{\vec{2}}-\epsilon_{\vec{4}}-\epsilon_{\vec{5}}-\epsilon_{\vec{6}}}\right) . \tag{H2b}$$

From Eq. (2.18) we obtain the exact relations

$$\Phi^{(1)}_{\vec{5}\vec{6}\vec{3}\vec{4}}-\Phi^{(1)}_{\vec{3}\vec{4}\vec{5}\vec{6}}=(\epsilon_{\vec{5}}+\epsilon_{\vec{6}}-\epsilon_{\vec{3}}-\epsilon_{\vec{4}})(1-x_{\vec{3}}x_{\vec{4}}x_{\vec{5}}x_{\vec{6}}) , \tag{H3a}$$

$$\Phi^{(8)}_{\vec{5}\vec{6}\vec{3}\vec{4}}-\Phi^{(7)}_{\vec{3}\vec{4}\vec{5}\vec{6}}=(\epsilon_{\vec{3}}+\epsilon_{\vec{4}}+\epsilon_{\vec{5}}+\epsilon_{\vec{6}})(x_{\vec{3}}x_{\vec{4}}-x_{\vec{5}}x_{\vec{6}}) , \tag{H3b}$$

$$\Phi^{(3)}_{\vec{2}\vec{5}\vec{4}\vec{6}}-\Phi^{(2)}_{\vec{4}\vec{6}\vec{2}\vec{5}}=(\epsilon_{\vec{4}}-\epsilon_{\vec{2}}-\epsilon_{\vec{5}}-\epsilon_{\vec{6}})(x_{\vec{6}}-x_{\vec{2}}x_{\vec{4}}x_{\vec{5}}) . \tag{H3c}$$

We shall only be concerned with the on-shell on-resonance case:

$$\tilde{\omega}=\epsilon_{\vec{1}}=\epsilon_{\vec{3}}+\epsilon_{\vec{4}}-\epsilon_{\vec{2}} . \tag{H4}$$

The Hermiticity properties we wish to establish may be written as

$$\Delta S\equiv S_{\vec{3}\vec{4}\vec{1}\vec{2}}-S_{\vec{1}\vec{2}\vec{3}\vec{4}}=0 , \tag{H5a}$$

$$\Delta U\equiv U_{\vec{3}\vec{4}\vec{1}\vec{2}}-U_{\vec{1}\vec{2}\vec{3}\vec{4}}=0 . \tag{H5b}$$

We have already seen that these relations are valid to lowest order in $1/S$:

$$\Delta\Phi\equiv\Phi^{(1)}_{\vec{3}\vec{4}\vec{1}\vec{2}}-\Phi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}}=0 . \tag{H5c}$$

In order to study these relations to the next order in $1/S$, we write

$$\Delta S-\Delta\Phi\equiv S_{\vec{3}\vec{4}\vec{1}\vec{2}}-S_{\vec{1}\vec{2}\vec{3}\vec{4}}-\Phi^{(1)}_{\vec{3}\vec{4}\vec{1}\vec{2}}+\Phi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}}$$
$$=(2NS)^{-1}\sum_{\vec{5},\vec{6}}l_{\vec{5}}^2 l_{\vec{6}}^2\delta_{\vec{k}}(\vec{k}_1+\vec{k}_2-\vec{k}_5-\vec{k}_6)\,[-\Phi^{(1)}_{\vec{1}\vec{2}\vec{5}\vec{6}}(1-x_{\vec{3}}x_{\vec{4}}x_{\vec{5}}x_{\vec{6}})+\Phi^{(1)}_{\vec{5}\vec{6}\vec{3}\vec{4}}(1-x_{\vec{1}}x_{\vec{2}}x_{\vec{5}}x_{\vec{6}})$$
$$+(\epsilon_{\vec{1}}+\epsilon_{\vec{2}}-\epsilon_{\vec{5}}-\epsilon_{\vec{6}})(1-x_{\vec{1}}x_{\vec{2}}x_{\vec{5}}x_{\vec{6}})(1-x_{\vec{3}}x_{\vec{4}}x_{\vec{5}}x_{\vec{6}})+\Phi^{(7)}_{\vec{1}\vec{2}\vec{5}\vec{6}}(x_{\vec{5}}x_{\vec{6}}-x_{\vec{3}}x_{\vec{4}})+\Phi^{(8)}_{\vec{5}\vec{6}\vec{3}\vec{4}}(x_{\vec{1}}x_{\vec{2}}-x_{\vec{5}}x_{\vec{6}})$$
$$+(\epsilon_{\vec{1}}+\epsilon_{\vec{2}}+\epsilon_{\vec{5}}+\epsilon_{\vec{6}})(x_{\vec{5}}x_{\vec{6}}-x_{\vec{3}}x_{\vec{4}})(x_{\vec{1}}x_{\vec{2}}-x_{\vec{5}}x_{\vec{6}})] , \tag{H6a}$$

$$\Delta U\equiv U_{\vec{3}\vec{4}\vec{1}\vec{2}}-U_{\vec{1}\vec{2}\vec{3}\vec{4}}=(NS)^{-1}\sum_{\vec{5},\vec{6}}l_{\vec{5}}^2 l_{\vec{6}}^2\delta_{\vec{k}}(\vec{k}_3+\vec{k}_6-\vec{k}_1-\vec{k}_5)\,[\Phi^{(3)}_{\vec{1}\vec{5}\vec{3}\vec{6}}(x_{\vec{4}}x_{\vec{5}}x_{\vec{2}}-x_{\vec{6}})+\Phi^{(2)}_{\vec{2}\vec{6}\vec{4}\vec{5}}(x_{\vec{6}}-x_{\vec{1}}x_{\vec{3}}x_{\vec{5}})$$
$$+(\epsilon_{\vec{4}}+\epsilon_{\vec{5}}+\epsilon_{\vec{6}}-\epsilon_{\vec{2}})(x_{\vec{4}}x_{\vec{5}}x_{\vec{2}}-x_{\vec{6}})(x_{\vec{6}}-x_{\vec{1}}x_{\vec{3}}x_{\vec{5}})+\Phi^{(2)}_{\vec{1}\vec{5}\vec{3}\vec{6}}(x_{\vec{5}}-x_{\vec{2}}x_{\vec{4}}x_{\vec{6}})+\Phi^{(3)}_{\vec{2}\vec{6}\vec{4}\vec{5}}(x_{\vec{1}}x_{\vec{3}}x_{\vec{6}}-x_{\vec{5}})$$

$$+(\epsilon_{\vec{2}}+\epsilon_{\vec{5}}+\epsilon_{\vec{6}}-\epsilon_{\vec{4}})(x_{\vec{5}}-x_{\vec{2}}x_{\vec{4}}x_{\vec{6}})(x_{\vec{1}}x_{\vec{3}}x_{\vec{6}}-x_{\vec{5}})]+(2NS)^{-1}\sum_{\vec{5},\vec{6}} l_{\vec{5}}^{2}\, l_{\vec{6}}^{2}\, \delta(\vec{k}_4+\vec{k}_6-\vec{k}_1-\vec{k}_5)$$

$$\times[\Phi^{(3)}_{\vec{1}\vec{5}\vec{4}\vec{6}}(x_{\vec{3}}x_{\vec{5}}x_{\vec{2}}-x_{\vec{6}})+\Phi^{(2)}_{\vec{2}\vec{6}\vec{3}\vec{5}}(x_{\vec{6}}-x_{\vec{1}}x_{\vec{4}}x_{\vec{5}})+(\epsilon_{\vec{3}}+\epsilon_{\vec{5}}+\epsilon_{\vec{6}}-\epsilon_{\vec{2}})(x_{\vec{3}}x_{\vec{5}}x_{\vec{2}}-x_{\vec{6}})(x_{\vec{6}}-x_{\vec{1}}x_{\vec{4}}x_{\vec{5}})$$

$$+\Phi^{(2)}_{\vec{1}\vec{5}\vec{4}\vec{6}}(x_{\vec{5}}-x_{\vec{2}}x_{\vec{3}}x_{\vec{6}})+\Phi^{(3)}_{\vec{2}\vec{6}\vec{3}\vec{5}}(x_{\vec{1}}x_{\vec{4}}x_{\vec{6}}-x_{\vec{5}})+(\epsilon_{\vec{2}}+\epsilon_{\vec{5}}+\epsilon_{\vec{6}}-\epsilon_{\vec{3}})(x_{\vec{5}}-x_{\vec{2}}x_{\vec{3}}x_{\vec{6}})(x_{\vec{1}}x_{\vec{4}}x_{\vec{6}}-x_{\vec{5}})]. \quad \text{(H6b)}$$

After some tedious algebra we find

$$\Delta S-\Delta\Phi=(2NS)^{-1}\sum_{\vec{5},\vec{6}}\delta_{\vec{k}}(\vec{k}_1+\vec{k}_2-\vec{k}_5-\vec{k}_6)\{-\gamma_{\vec{1}-\vec{5}}[(x_{\vec{1}}+x_{\vec{2}}x_{\vec{3}}x_{\vec{4}})l_{\vec{5}}^2x_{\vec{5}}+(x_{\vec{2}}+x_{\vec{1}}x_{\vec{3}}x_{\vec{4}})l_{\vec{6}}^2x_{\vec{6}}]$$

$$-\gamma_{\vec{2}-\vec{5}}[(x_{\vec{2}}+x_{\vec{1}}x_{\vec{3}}x_{\vec{4}})l_{\vec{5}}^2x_{\vec{5}}+(x_{\vec{1}}+x_{\vec{2}}x_{\vec{3}}x_{\vec{4}})l_{\vec{6}}^2x_{\vec{6}}]+\gamma_{\vec{5}-\vec{3}}[(x_{\vec{3}}+x_{\vec{1}}x_{\vec{2}}x_{\vec{4}})l_{\vec{5}}^2x_{\vec{5}}+(x_{\vec{4}}+x_{\vec{1}}x_{\vec{2}}x_{\vec{3}})l_{\vec{6}}^2x_{\vec{6}}]$$

$$+\gamma_{\vec{5}-\vec{4}}[(x_{\vec{4}}+x_{\vec{1}}x_{\vec{2}}x_{\vec{3}})l_{\vec{5}}^2x_{\vec{5}}+(x_{\vec{3}}+x_{\vec{1}}x_{\vec{2}}x_{\vec{4}})l_{\vec{6}}^2x_{\vec{6}}]\}+(2NS)^{-1}\sum_{\vec{5},\vec{6}} l_{\vec{5}}^2l_{\vec{6}}^2\delta(k_{\vec{1}}+k_{\vec{2}}-k_{\vec{5}}-k_{\vec{6}})$$

$$\times\{[2-\epsilon_{\vec{1}}-\epsilon_{\vec{2}}+x_{\vec{1}}x_{\vec{2}}x_{\vec{5}}x_{\vec{6}}(2+\epsilon_{\vec{1}}+\epsilon_{\vec{2}})](1-x_{\vec{3}}x_{\vec{4}}x_{\vec{5}}x_{\vec{6}})[-2+\epsilon_{\vec{5}}+\epsilon_{\vec{6}}-x_{\vec{3}}x_{\vec{4}}x_{\vec{5}}x_{\vec{6}}(2+\epsilon_{\vec{5}}+\epsilon_{\vec{6}})](1-x_{\vec{1}}x_{\vec{2}}x_{\vec{5}}x_{\vec{6}})$$

$$+(\epsilon_{\vec{1}}+\epsilon_{\vec{2}}-\epsilon_{\vec{5}}-\epsilon_{\vec{6}})(1-x_{\vec{1}}x_{\vec{2}}x_{\vec{5}}x_{\vec{6}})(1-x_{\vec{3}}x_{\vec{4}}x_{\vec{5}}x_{\vec{6}})+[x_{\vec{1}}x_{\vec{2}}(2+\epsilon_{\vec{1}}+\epsilon_{\vec{2}})+x_{\vec{5}}x_{\vec{6}}(2-\epsilon_{\vec{1}}-\epsilon_{\vec{2}})](x_{\vec{3}}x_{\vec{4}}-x_{\vec{5}}x_{\vec{6}})$$

$$+[x_{\vec{5}}x_{\vec{6}}(2+\epsilon_{\vec{5}}+\epsilon_{\vec{6}})+x_{\vec{3}}x_{\vec{4}}(2-\epsilon_{\vec{5}}-\epsilon_{\vec{6}})](x_{\vec{5}}x_{\vec{6}}-x_{\vec{1}}x_{\vec{2}})+(\epsilon_{\vec{1}}+\epsilon_{\vec{2}}+\epsilon_{\vec{5}}+\epsilon_{\vec{6}})(x_{\vec{5}}x_{\vec{6}}-x_{\vec{3}}x_{\vec{4}})(x_{\vec{1}}x_{\vec{2}}-x_{\vec{5}}x_{\vec{6}})\}, \quad \text{(H7)}$$

where we have used $\gamma_{\vec{p}}x_{\vec{p}}=1-\epsilon_{\vec{p}}$ and $\gamma_{\vec{p}}/x_{\vec{p}}=1+\epsilon_{\vec{p}}$. In the second summation in Eq. (H7) the summand vanishes, and the first one can be evaluated using the formula

$$\sum_{\vec{u}}\gamma_{\vec{u}-\vec{t}}f(\vec{u})=\gamma_{\vec{t}}\sum_{\vec{u}}\gamma_{\vec{u}}f(\vec{u}) \quad \text{(H8)}$$

for functions $f(\vec{u})$ which are invariant under the cubic symmetry group. In this way we find that when $\epsilon_{\vec{1}}+\epsilon_{\vec{2}}=\epsilon_{\vec{3}}+\epsilon_{\vec{4}}$, and $k_{\vec{1}}+k_{\vec{2}}=k_{\vec{3}}+k_{\vec{4}}$, the first summation in Eq. (H7) vanishes also, and thus, on resonance $\Delta S-\Delta\Phi=0$. But since Φ is Hermitian on resonance, i.e., since $\Delta\Phi=0$, it follows that $\Delta S=0$, i.e., S is Hermitian on resonance. By an entirely similar calculation, we have also verified that $\Delta U=0$ on resonance.

Thus we have shown that insofar as the terms involving the scattering of four α magnons are concerned, the dressed vertices are Hermitian on resonance up to order $1/S$. As is discussed in Sec. V, this result is just the property needed to ensure that the lowest Born results do not suffer anomalous temperature renormalizations when higher-order perturbation terms are included.

2. Ferromagnet at Low Magnon Density

We shall show that the effective interactions at low magnon density in a ferromagnet are Hermitian on resonance. In this low-density regime, it is possible to express the static and dynamic quantities in terms of the two-spin-wave t matrix, which thus plays the role of the energy-dependent dressed vertex for the ferromagnet. The t matrix obeys the equation[6-8]

$$t(z;\vec{K};\vec{k}_1,\vec{k}_2)=V_{\vec{K}}(\vec{k}_1,\vec{k}_2)+N^{-1}\sum_{\vec{k}_3}V_{\vec{K}}(\vec{k}_1,\vec{k}_3)\times\frac{1+n(\frac{1}{2}\vec{K}+\vec{k}_3)+n(\frac{1}{2}\vec{K}-\vec{k}_3)}{\hbar z-\mathcal{E}_{\vec{K}}(\vec{k}_3)}\,t(z;\vec{K};\vec{k}_3,\vec{k}_2), \quad \text{(H9)}$$

where we shall use the conventions of Ref. 8, so that the potential can be written in the form

$$V_{\vec{K}}(\vec{k}_1,\vec{k}_2)=(2S)^{-1}[E(\vec{k}_1-\vec{k}_2)+E(\vec{k}_1+\vec{k}_2)-\mathcal{E}_{\vec{K}}(\vec{k}_1)], \quad \text{(H10)}$$

where $E(\vec{k})$ is the unperturbed magnon energy:

$$E(\vec{k})=2JzS(1-\gamma_{\vec{k}}), \quad \text{(H11)}$$

and we have also introduced the two-particle propagation energy as

$$\mathcal{E}_{\vec{K}}(\vec{k})=E(\tfrac{1}{2}\vec{K}+\vec{k})+E(\tfrac{1}{2}\vec{K}-\vec{k}). \quad \text{(H12)}$$

For convenience, we introduce the following matrix notation:

$$t_{\vec{1}\vec{2}}\equiv t(z;\vec{K};\vec{k}_1,\vec{k}_2), \quad \text{(H13a)}$$

$$V_{\vec{1}\vec{2}}\equiv V_{\vec{K}}(\vec{k}_1,\vec{k}_2), \quad \text{(H13b)}$$

$$D_{\vec{1}\vec{2}}\equiv\delta_{\vec{k}_1,\vec{k}_2}\left(\frac{1+n(\frac{1}{2}\vec{K}+\vec{k}_1)+n(\frac{1}{2}\vec{K}-\vec{k}_1)}{\hbar z-\mathcal{E}_{\vec{K}}(\vec{k}_1)}\right)\equiv D_{\vec{1}}. \quad \text{(H13c)}$$

Using this notation, we write Eq. (H9) as

$$\underline{\mathrm{t}}=\underline{\mathrm{V}}+\underline{\mathrm{VDt}} \quad \text{(H14a)}$$

or, equivalently, as

$$\underline{\mathrm{t}} = \underline{\mathrm{V}} + \underline{\mathrm{tDV}} \quad , \tag{H14b}$$

from which we see that the transpose of $\underline{\mathrm{t}}$, denoted $\underline{\tilde{\mathrm{t}}}$, $(\tilde{t}_{\vec{1}\vec{2}} = t_{\vec{2}\vec{1}})$ satisfies

$$\underline{\tilde{\mathrm{t}}} = \underline{\tilde{\mathrm{V}}} + \underline{\tilde{\mathrm{V}}\mathrm{D}\tilde{\mathrm{t}}} \quad , \tag{H14c}$$

where $\tilde{V}_{\vec{1}\vec{2}} = V_{\vec{2}\vec{1}}$. Since the potential in Eq. (H13b) is real, the t matrix satisfies

$$\underline{\mathrm{t}}(z)^* = \underline{\mathrm{t}}(z^*) \; . \tag{H15}$$

For a Hermitian potential, it is easy to show that on resonance the t matrix satisfies the Hermiticity relation

$$t(\omega - i\delta; \vec{K}; \vec{k}_1, \vec{k}_2)^* = t(\omega + i\delta; \vec{K}; \vec{k}_2, \vec{k}_1) \; . \tag{H16}$$

We wish to show that this relation still holds, at least to lowest order in the density of magnons, for the non-Hermitian potential of Eq. (H10), when $\hbar\omega = \epsilon_{\vec{K}}(\vec{k}_1) = \epsilon_{\vec{K}}(\vec{k}_2)$. At low magnon density, we may replace the propagator $\underline{\mathrm{D}}$ by its zero temperature value $\underline{\mathrm{D}}^0$, where

$$D^0_{\vec{1}\vec{2}} = \delta_{\vec{k}_1, \vec{k}_2}[\hbar z - \mathcal{E}_{\vec{K}}(\vec{k}_2)]^{-1} \equiv D^0_{\vec{1}} \; . \tag{H17}$$

Thus the zero-temperature t matrix, denoted $\underline{t}^0$, is given by

$$\underline{\mathrm{t}}^0 = \underline{\mathrm{V}} + \underline{\mathrm{VD}^0\mathrm{t}^0} \; . \tag{H18}$$

In this approximation, Eq. (H16) may be written as

$$t^0_{\vec{1}\vec{2}} - \tilde{t}^0_{\vec{1}\vec{2}} = 0 \quad \text{for} \quad \hbar z = \mathcal{E}_{\vec{K}}(\vec{k}_1) - i\delta = \mathcal{E}_{\vec{K}}(\vec{k}_2) - i\delta \; , \tag{H19}$$

where we have used Eq. (H15).

We shall demonstrate Eq. (H19) by expanding the t matrix in powers of the potential V and proving that Eq. (H19) holds to all orders in V. Using Eqs. (H14a) and (H14c), we write Eq. (H19) to nth order in V as

$$V_{\vec{a}\vec{b}} - \tilde{V}_{\vec{a}\vec{b}} = 0, \quad n = 1 \; ; \tag{H20a}$$

$$V_{\vec{a}\vec{1}} D^0_{\vec{1}} V_{\vec{1}\vec{2}} D^0_{\vec{2}} \cdots D^0_{\vec{n}-\vec{1}} V_{\vec{n}-\vec{1},\vec{b}} - \tilde{V}_{\vec{a}\vec{1}} D^0_{\vec{1}} \tilde{V}_{\vec{1}\vec{2}} D^0_{\vec{2}} \cdots D^0_{\vec{n}-\vec{1}} \tilde{V}_{\vec{n}-\vec{1},\vec{b}} = 0, \quad n \geq 2 \; ; \tag{H20b}$$

where $\vec{a}$ and $\vec{b}$ denote the initial and final states with relative momenta $\vec{k}_a$ and $\vec{k}_b$, respectively. On resonance we have

$$D^0_{\vec{r}} = [\mathcal{E}_{\vec{K}}(\vec{k}_a) - \mathcal{E}_{\vec{K}}(\vec{k}_r) - i\delta]^{-1} = [\mathcal{E}_{\vec{K}}(\vec{k}_b) - \mathcal{E}_{\vec{K}}(\vec{k}_r) - i\delta]^{-1} \; , \tag{H21}$$

since for the t matrix on resonance

$$\mathcal{E}_{\vec{K}}(\vec{k}_a) = \mathcal{E}_{\vec{K}}(\vec{k}_b) \; . \tag{H22}$$

We now verify Eqs. (H20a) and (H20b). We shall make repeated use of the relation

$$\sum_{\vec{k}_1} V_{\vec{K}}(\vec{k}_1, \vec{k}_2) \equiv \sum_{\vec{1}} V_{\vec{1}\vec{2}} = \sum_{\vec{1}} \tilde{V}_{\vec{2}\vec{1}} = 0 \; , \tag{H23}$$

which follows from Eq. (H10) when

$$\textstyle\sum_{\vec{k}} \gamma_{\vec{k}} = 0 \tag{H24}$$

is used. Also note that using Eq. (H10) we may write

$$2S(V_{\vec{1}\vec{2}} - \tilde{V}_{\vec{1}\vec{2}}) = \mathcal{E}_{\vec{K}}(\vec{k}_2) - \mathcal{E}_{\vec{K}}(\vec{k}_1) \; . \tag{H25}$$

Using Eqs. (H22) and (H25), it is clear that Eq. (H20a) is valid. To establish Eq. (H20b), we define

$$\underline{\mathrm{T}}^{n0} = \underline{(\mathrm{VD}^0)}^{n-1}\underline{\mathrm{V}} \; , \tag{H26a}$$

$$\underline{\mathrm{T}}^{nr} = \underline{(\tilde{\mathrm{V}}\mathrm{D}^0)}^{r}\underline{(\mathrm{VD}^0)}^{n-1-r}\underline{\mathrm{V}}, \quad 1 \leq r \leq n-1 \tag{H26b}$$

$$\underline{\mathrm{T}}^{nn} = \underline{(\tilde{\mathrm{V}}\mathrm{D}^0)}^{n-1}\underline{\tilde{\mathrm{V}}} \; , \tag{H26c}$$

and we wish to show that

$$T^{n0}_{\vec{a}\vec{b}} = T^{nn}_{\vec{a}\vec{b}} \tag{H27}$$

or, equivalently, that

$$T^{ns}_{\vec{a}\vec{b}} = T^{n,s-1}_{\vec{a}\vec{b}} \; , \quad s = 1, 2, \ldots, n \; . \tag{H28}$$

In other words, we shall show that $\underline{\tilde{\mathrm{t}}}$ can be obtained from $\underline{\mathrm{t}}$ by successively replacing the $\underline{\mathrm{V}}$'s in Eq. (H20b) by $\underline{\tilde{\mathrm{V}}}$'s starting from the left. We first show that $T^{n1}_{\vec{a}\vec{b}} = T^{n0}_{\vec{a}\vec{b}}$. We have, for $n \geq 2$,

$$T^{n1}_{\vec{a}\vec{b}} - T^{n0}_{\vec{a}\vec{b}} = \sum_{\vec{1},\vec{2},\ldots,\vec{n}-\vec{1}} (V_{\vec{a}\vec{1}} - \tilde{V}_{\vec{a}\vec{1}}) D^0_{\vec{1}} \cdots D^0_{\vec{n}-\vec{1}} V_{\vec{n}-\vec{1},\vec{b}} \; , \tag{H29}$$

but from Eqs. (H25), (H22), and (H21), we see that

$$(V_{\vec{a}\vec{1}} - \tilde{V}_{\vec{a}\vec{1}}) D^0_{\vec{1}} = -(2S)^{-1} \; . \tag{H30}$$

Taking account of Eq. (H23) and substituting Eq. (H30) into Eq. (H29), we see that the sum over $\vec{k}_1$ vanishes, so that

$$T^{n1}_{\vec{a}\vec{b}} = T^{n0}_{\vec{a}\vec{b}} \; . \tag{H31}$$

Next, for $n \geq 3$, consider

$$T^{n2}_{\vec{a}\vec{b}} - T^{n1}_{\vec{a}\vec{b}} = -\sum_{\vec{1},\vec{2},\ldots,\vec{n}-\vec{1}} \tilde{V}_{\vec{a}\vec{1}} D^0_{\vec{1}} (V_{\vec{1}\vec{2}} - \tilde{V}_{\vec{1}\vec{2}}) \cdots D^0_{\vec{n}-\vec{1}} V_{\vec{n}-\vec{1},\vec{b}} \; . \tag{H32}$$

Using Eqs. (H21) and (H25), we may write

$$\begin{aligned} D^0_{\vec{1}}(V_{\vec{1}\vec{2}} - \tilde{V}_{\vec{1}\vec{2}}) D^0_{\vec{2}} &= (2S)^{-1} D^0_{\vec{1}} [\mathcal{E}_{\vec{K}}(\mathrm{k}_2) - \mathcal{E}_{\vec{K}}(\vec{k}_1)] D^0_{\vec{2}} \\ &= (2S)^{-1} D^0_{\vec{1}} [\mathcal{E}_{\vec{K}}(\vec{k}_2) - \mathcal{E}_{\vec{K}}(\vec{k}_a)] D^0_{\vec{2}} \\ &\quad - (2S)^{-1} D^0_{\vec{1}} [\mathcal{E}_{\vec{K}}(\vec{k}_1) - \mathcal{E}_{\vec{K}}(\vec{k}_a)] D^0_{\vec{2}} \\ &= -(2S)^{-1}(D^0_{\vec{1}} - D^0_{\vec{2}}) \; . \end{aligned} \tag{H33}$$

Then may write

$$T^{n2}_{\vec{a}\vec{b}} - T^{n1}_{\vec{a}\vec{b}} = -(2S)^{-1} \sum_{\vec{1},\vec{2},\ldots,\vec{n}-\vec{1}} \tilde{V}_{\vec{a}\vec{1}} D^0_{\vec{2}} \cdots V_{\vec{n}-\vec{1},\vec{b}} + (2S)^{-1} \sum_{\vec{1},\vec{2},\ldots,\vec{n}-\vec{1}} \tilde{V}_{\vec{a}\vec{1}} D^0_{\vec{1}} \cdots V_{\vec{n}-\vec{1},\vec{b}} \; , \tag{H34}$$

where in the last term $V_{\vec{2}\vec{3}} \cdots V_{\vec{n}-\vec{1},\vec{b}}$ becomes $V_{\vec{2}\vec{b}}$ for $n = 3$. In any case, the sum over $\vec{k}_1$ in the first term and that over $\vec{k}_2$ in the second term vanish because of Eq. (H23). Thus we have that

$$T^{n2}_{\vec{a}\vec{b}} = T^{n1}_{\vec{a}\vec{b}} \; . \tag{H35}$$

This process can be continued indefinitely and establishes Eq. (H28). Thus Eq. (H19) is verified to all orders in V, and we conclude that the effective interactions between spin waves in the Dyson-Maleev representation are Hermitian on resonance at low temperatures.

Let us make a few comments about this result. First of all, it has only been proved to lowest order in the density of spin waves. A treatment of the next order in the density of spin waves $[\sim (kT/JS)^{3/2}]$ would be extremely difficult. Second, observe that the proof depends essentially on the interplay between the interaction potential and the energy denominators and hence cannot be expected to remain valid when damping of intermediate states is included.

APPENDIX I: HYDRODYNAMICS AT LOW TEMPERATURES TO ALL ORDERS IN $1/z$

1. Spin-Correlation Functions in Hydrodynamic Regime at Low Temperatures

In this Appendix, we shall obtain exact forms for the spin-correlation functions $C_Q^{+-}(\vec{k},\omega)$ and $C_S^{+-}(\vec{k},\omega)$ in the low-temperature long-wavelength limits, correct to all orders in $1/z$ or $1/S$.[88] When higher-order effects in $1/z$ or $1/S$ are taken into account, it is no longer helpful to introduce a spin self-energy. Instead, we shall calculate the spin-correlation functions directly. Of course, actual numerical evaluation of the vertex functions to all orders in $1/z$ or $1/S$ is essentially out of the question. However, since we have established the long-wavelength behavior of these functions and the general relations among them, we may parametrize them in a way similar to the Landau theory of a Fermi liquid.[96,63]

To do this, we shall use the relation between the spin and boson Green's functions

$$\underline{\mathcal{G}} = (\underline{1} + \underline{\Lambda})\underline{G} \;, \tag{I1}$$

where the boson Green's function is obtained from the self-energy by

$$\underline{G} = (\underline{G}_0^{-1} - \underline{\Sigma})^{-1} \; . \tag{I2}$$

Finally, the spin-correlation functions are given in terms of the spin Green's functions by [see Eqs. (7.22) and (2.36)]

$$\begin{aligned} C_S^{+-}(\vec{k},\omega) &= (2\hbar^2 S k_B T \epsilon_{\vec{k}}/\omega) \\ &\quad\times \mathrm{Im}[\mathcal{G}_{\alpha\alpha}(\vec{k},\omega) + \mathcal{G}_{\beta\beta}(\vec{k},\omega) \\ &\quad + \mathcal{G}_{\alpha\beta}(\vec{k},\omega) + \mathcal{G}_{\beta\alpha}(\vec{k},\omega)] \;, \end{aligned} \tag{I3a}$$

$$\begin{aligned} C_Q^{+-}(\vec{k},\omega) &= (8\hbar^2 S k_B T/\epsilon_{\vec{k}}\omega) \\ &\quad\times \mathrm{Im}[\mathcal{G}_{\alpha\alpha}(\vec{k},\omega) + \mathcal{G}_{\beta\beta}(\vec{k},\omega) \\ &\quad - \mathcal{G}_{\alpha\beta}(\vec{k},\omega) - \mathcal{G}_{\beta\alpha}(\vec{k},\omega)] \; . \end{aligned} \tag{I3b}$$

Having constructed the spin-correlation functions, we shall then be able to compare our results with the predictions of hydrodynamics.[13]

We summarize the results of Appendix G 4, where the following asymptotic forms at long wavelengths and low temperatures ($\epsilon_{\vec{k}} \ll \tau^3 \ll 1$) were obtained:

$$\begin{aligned} \Sigma_{\alpha\alpha}(\vec{k},\omega - iO^+) &= -\Sigma_{\beta\alpha}(\vec{k},\omega - iO^+) = -\Sigma_{\alpha\beta}(\vec{k},-\omega + iO^+) \\ &= \Sigma_{\beta\beta}(\vec{k},-\omega+iO^+) = \hbar\omega_E\epsilon_{\vec{k}}(C_3 + C_4\rho) \\ &\quad + i\hbar\omega_E\epsilon_{\vec{k}}^2(C_1\rho + C_2\rho^2) \end{aligned} \tag{I4}$$

and

$$\begin{aligned} 1 + \Lambda_{\alpha\alpha}(\vec{k},\omega - iO^+) &= 1 + \Lambda_{\beta\beta}(\vec{k},-\omega + iO^+) \\ &= \sigma + C_6 + iC_5\epsilon_{\vec{k}}\rho \;, \end{aligned} \tag{I5a}$$

$$\begin{aligned} \Lambda_{\alpha\beta}(\vec{k},\omega - iO^+) &= \Lambda_{\beta\alpha}(\vec{k},-\omega + iO^+) \\ &= C_6 + iC_5\epsilon_{\vec{k}}\rho \; . \end{aligned} \tag{I5b}$$

Here $\sigma = \langle S_a^z\rangle/S$, $\rho = \hbar\omega/E_{\vec{k}} = \omega/\omega_E\epsilon_{\vec{k}} = \bar{\omega}/\epsilon_{\vec{k}}$, and $C_1, \ldots, C_6$ are parameters independent of frequency and momentum. Inserting these results into Eq. (I2) we determine the boson Green's function as a 2×2 matrix;

$$\underline{G} = \hbar\omega_E\epsilon_{\vec{k}}\Delta^{-1}\begin{pmatrix} -\rho(1-C_4) - (1+C_3) - i\epsilon_{\vec{k}}(C_1\rho - C_2\rho^2) & -C_3 + C_4\rho - i\epsilon_{\vec{k}}(C_1\rho - C_2\rho^2) \\ -C_3 - C_4\rho - i\epsilon_{\vec{k}}(C_1\rho + C_2\rho^2) & \rho(1-C_4) - (1+C_3) - i\epsilon_{\vec{k}}(C_1\rho + C_2\rho^2)\end{pmatrix}, \tag{I6}$$

where $\Delta = \det\underline{G}^{-1}$, which to leading order in $\epsilon_{\vec{k}}$ is given as

$$\Delta = (\hbar\omega_E\epsilon_{\vec{k}})^2(1-2C_4)[\rho_0^2 - \rho^2 + 2i\rho\,\epsilon_{\vec{k}}(C_1 + C_2\rho^2)(1-2C_4)^{-1}], \tag{I7}$$

where

$$\rho_0^2 = (1+2C_3)/(1-2C_4) \; . \tag{I8}$$

The spin-wave velocity c is determined from the poles of $\underline{G}$, which are found by setting $\mathrm{Re}\Delta = 0$, which yields

$$c = \tfrac{1}{2}\omega_E \rho_0 \; . \tag{I9}$$

The next step in constructing the correlation functions is to evaluate the spin Green's functions, $\mathcal{G}(\vec{k}, \omega - iO^+)$, via Eq. (I1) with the boson Green's function of Eq. (I6) and the vertex function $\underline{\Lambda}$ as given in Eq. (I5). For instance, these evaluations yield

$$\mathrm{Im}(\mathcal{G}_{\alpha\alpha} + \mathcal{G}_{\beta\beta} + \mathcal{G}_{\alpha\beta} + \mathcal{G}_{\beta\alpha}) = |\Delta|^{-2}\,\mathrm{Im}\{\hbar\omega_E \Delta^*(\sigma + 2C_6 + 2iC_5\epsilon_{\vec{k}}\,\rho)[-2\epsilon_{\vec{k}}(1+2C_3) - 4i\epsilon_{\vec{k}}^2 C_1\rho]\} \; . \tag{I10}$$

Note that to lowest order in $\epsilon_{\vec{k}}$, we can write

$$|\Delta|^2 = \hbar^4(1-2C_4)^2[(\omega - ck)^2 + (\tfrac{1}{2}D'k^2)^2][(\omega + ck)^2 + (\tfrac{1}{2}D'k^2)^2] \; , \tag{I11}$$

where

$$D' = \tfrac{1}{2}\omega_E(C_1 + C_2\rho_0^2)(1-2C_4)^{-1} \; . \tag{I12}$$

In this way we express the total-spin-correlation function as

$$C_S^{+-}(\vec{k}, \omega) = \left(\frac{2\hbar S k_B T k^2}{1-2C_4}\right) \frac{(\omega^2 - c^2k^2)[C_5(1+2C_3) + (\sigma + 2C_6)(C_1 + C_2\rho_0^2)] + c^2k^2[(\sigma+2C_6)(C_1+C_2\rho_0^2)]}{[(\omega-ck)^2 + (\frac{1}{2}D'k^2)^2][(\omega+ck)^2 + (\frac{1}{2}D'k^2)^2]} \tag{I13a}$$

and the staggered-spin-correlation function as

$$C_Q^{+-}(\vec{k}, \omega) = \left(\frac{32\hbar S\sigma k_B T}{(1-2C_4)(1+2C_3)}\right) \frac{[(\omega^2 - c^2k^2)C_2\rho_0^2 + c^2k^2(C_1 + C_2\rho_0^2)]}{[(\omega-ck)^2 + (\frac{1}{2}D'k^2)^2][(\omega+ck)^2 + (\frac{1}{2}D'k^2)^2]} \; . \tag{I13b}$$

Note that to lowest order in $1/z$, one has $\sigma = 1$, $C_4 = C_3 = C_6 = 0$, $C_1 = C_1^0$, $C_2 = C_2^0$, and $C_5 = -C_2^0$, as found in the lowest Born approximation, and, consequently, we recover the results of Sec. VII.

These spin-correlation functions are indeed of the hydrodynamic form given in Eq. (7.26), and from the poles of the correlation functions we may make the identification

$$D = D' \; , \tag{I14a}$$

and also we confirm that the microscopic and hydrodynamic spin-wave velocities coincide. It is also clear that we should set

$$N_0 = 2\hbar S\sigma \; . \tag{I14b}$$

Next, we determine the transport coefficients. Note, however, that according to the hydrodynamic theory the parameters in the correlation function must obey Eqs. (7.27b) and (7.28). In order for the correlation functions of Eq. (I13) to be consistent with those relations, we must have

$$\sigma C_2 + C_5 = 0 \; , \tag{I15}$$

$$\sigma C_4 + C_6 = 0 \; . \tag{I16}$$

Assuming these relations to hold, we then find by comparison with the hydrodynamic results, that the transport coefficients and thermodynamic parameters are given as

$$\rho_s = \tfrac{1}{2}c\hbar S\sigma\rho_0(1-2C_4) \; , \tag{I17a}$$

$$K_\perp = \tfrac{1}{2}C_1\hbar S\sigma \; , \tag{I17b}$$

$$\chi = (\hbar S/2c)\sigma\rho_0(1-2C_4) \; , \tag{I17c}$$

$$\zeta = 2C_2[\sigma\hbar S(1-2C_4)^2]^{-1} \; , \tag{I17d}$$

If we could derive the equalities (I15) and (I16), we would then have given a complete microscopic derivation of hydrodynamics in the low-temperature limit. We have not been able to prove these equalities, in general. However, in Appendix I 2 we show that they hold to order S^{-2}, at least for a selected subset of diagrams.

2. Verification of Hydrodynamic Relations

In this section, we shall check that the relations given in Eqs. (I15) and (I16) remain valid when the first-order dressed vertices of Appendix H are used for the imaginary parts of the vertex functions, and the real parts are calculated in second-order perturbation theory. Although this check is by no means a complete proof, it is a nontrivial test of hydrodynamics and of our formalism.

First we consider Eq. (I15). We shall study to order S^{-2} the quantity Q_A which is given as

$$Q_A = \Sigma''_{\alpha\alpha}(\vec{k}, \omega) + \hbar\omega_E\sigma^{-1}\epsilon_k(\rho - \rho_0)\Lambda''_{\alpha\alpha}(\vec{k}, \omega) \; . \tag{I18}$$

This quantity is a generalization of the spin self-energy [e.g., see Eq. (7.12)], but we have been unable to establish any sound physical interpretation along these lines. We shall rearrange the expression for Q_A into the form of Eq. (5.9), but where the matrix elements involve products of the form Φ^2, rather than $\Phi\tilde{\Phi}$. Then we may conclude that these matrix elements are of order $\epsilon_{\vec{k}}/\epsilon_{\vec{3}}$, and that there are no terms of order $\rho\epsilon_{\vec{k}}/\epsilon_{\vec{3}}$. Then Q_A is explicitly of order $\rho\epsilon_{\vec{k}}^2 \sim \omega E_{\vec{k}}$: There are no terms of order $\omega^2 \sim \rho^2\epsilon_{\vec{k}}^2$. But using Eqs. (G43) and (G46), this means that $C_2 + \sigma^{-1}C_5 = 0$, which is Eq. (I15), as desired.

We shall study contributions to these quantities which involve dressed vertices describing the collision of four α magnons. These contributions are of the form of Eq. (5.9) with $\mathfrak{M}_{22}$ replaced by

$$M_{22}^A(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) = 8l_{\vec{1}}^2 l_{\vec{2}}^2 l_{\vec{3}}^2 l_{\vec{4}}^2 (2R^{(1)}_{\vec{3}\vec{4}\vec{1}\vec{2}}\bar{R}^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}} - S_{\vec{3}\vec{4}\vec{1}\vec{2}}\bar{S}_{\vec{1}\vec{2}\vec{3}\vec{4}}) \;, \tag{I19}$$

where the dressed vertices $\bar{S}$ and $\bar{U}$ are calculated according to the same rules as S and U, but with the outgoing vertex associated with the coefficient

$$\Psi^{(i)} = \Phi^{(i)} + \sigma^{-1}\epsilon_{\vec{k}}(\rho-\rho_0)\hat{\Phi}^{(i)} \;, \tag{I20}$$

and $\bar{R} = \bar{S} + \bar{U}$. Thus we write schematically

$$\Sigma''_{\alpha\alpha} + \hbar\omega_E\sigma^{-1}\epsilon_{\vec{k}}(\rho-\rho_0)\Lambda''_{\alpha\alpha} \sim M_{22}^A \;. \tag{I21}$$

We wish to show that $Q_A = Q_B$, where

$$Q_B \sim M_{22}^B \;, \tag{I22}$$

i.e., Q_B is formed from Eq. (5.9) with $\mathfrak{M}_{22}$ replaced by M_{22}^B given by

$$M_{22}^B(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) = 8l_{\vec{1}}^2 l_{\vec{2}}^2 l_{\vec{3}}^2 l_{\vec{4}}^2 [2(R^{(1)}_{\vec{3}\vec{4}\vec{1}\vec{2}})^2 - (S_{\vec{3}\vec{4}\vec{1}\vec{2}})^2] \;. \tag{I23}$$

Since M_{22}^B is of order $\epsilon_{\vec{k}}$, Q_B is of order $\rho\epsilon_{\vec{k}}^2$ and if we show that Q_B and Q_A are equal, then we may conclude that Q_A is also of order $\rho\epsilon_{\vec{k}}^2$. As mentioned above, this implies that Eq. (I15) is valid.

Since we consider only contributions up to order S^{-2}, we may use the expressions for S and U which are given in Eqs. (H1) and (H2). The expressions for $\bar{S}$ and $\bar{U}$ are the same as those for S and U in Eqs. (H1) and (H2), except that in each case Φ's, involving $\vec{k}_3$ should be replaced by the corresponding Ψ's. Note also that to first order in S^{-1} we have

$$S_{\vec{1}\vec{2}\vec{3}\vec{4}} = \Phi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \;, \tag{I24a}$$

$$U_{\vec{1}\vec{2}\vec{3}\vec{4}} = 0 \;. \tag{I24b}$$

It is clear that Q_A and Q_B will be equal if M_{22}^A and M_{22}^B are equal on shell, i.e., for $\tilde{\omega} = \rho_0(\epsilon_{\vec{3}} + \epsilon_{\vec{4}} - \epsilon_{\vec{2}})$. Note that in this calculation it is necessary to use the renormalized energies, $\rho_0\epsilon_{\vec{k}}$ in the propagators (i.e., in the δ function). Using the aforementioned forms for S, $\bar{S}$, U, and $\bar{U}$, and also taking note of Eq. (I24), we find that to order S^{-2}, M_{22}^A and M_{22}^B are equal if

$$(\bar{S}_{\vec{1}\vec{2}\vec{3}\vec{4}} - S_{\vec{3}\vec{4}\vec{1}\vec{2}} + 2\bar{U}_{\vec{1}\vec{2}\vec{3}\vec{4}} - 2U_{\vec{3}\vec{4}\vec{1}\vec{2}})$$

$$\times\,\delta[\rho\epsilon_{\vec{1}} + \rho_0(\epsilon_{\vec{2}} - \epsilon_{\vec{3}} - \epsilon_{\vec{4}})] = 0 \;, \tag{I25}$$

where $\vec{k}_1 = \vec{k}$, and

$$\omega = \omega_E\rho\epsilon_1 \equiv \omega_E\rho_0\hat{\omega} \;, \tag{I26}$$

so that

$$\delta[\rho\epsilon_{\vec{1}} + \rho_0(\epsilon_{\vec{2}} - \epsilon_{\vec{3}} - \epsilon_{\vec{4}})] = \rho_0^{-1}\delta(\hat{\omega} + \epsilon_{\vec{2}} - \epsilon_{\vec{3}} - \epsilon_{\vec{4}}) \tag{I27a}$$

and

$$\rho\epsilon_{\vec{1}} - \rho_0\epsilon_{\vec{1}} = \rho_0(\hat{\omega} - \epsilon_{\vec{1}}) \;. \tag{I27b}$$

We use these relations to write Eq. (I25) in terms of the unperturbed energies and find that it will be valid if

$$(\Phi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}} + (\hat{\omega} - \epsilon_{\vec{1}})\hat{\Phi}^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}} - \Phi^{(1)}_{\vec{3}\vec{4}\vec{1}\vec{2}})\,\delta(\hat{\omega} + \epsilon_{\vec{2}} - \epsilon_{\vec{3}} - \epsilon_{\vec{4}}) = 0 \tag{I28}$$

and

$$\{(\bar{S}_{\vec{1}\vec{2}\vec{3}\vec{4}} - \Psi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}}) - (S_{\vec{3}\vec{4}\vec{1}\vec{2}} - \Phi^{(1)}_{\vec{3}\vec{4}\vec{1}\vec{2}}) + 2(\bar{U}_{\vec{1}\vec{2}\vec{3}\vec{4}} - U_{\vec{3}\vec{4}\vec{1}\vec{2}})$$

$$+[(\rho_0/\sigma)^{(1)} - 1](\hat{\omega} - \epsilon_{\vec{1}})\hat{\Phi}^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}}\}$$

$$\times\,\delta(\hat{\omega} + \epsilon_{\vec{2}} - \epsilon_{\vec{3}} - \epsilon_{\vec{4}}) = 0 \;, \tag{I29}$$

where $(\rho_0/\sigma)^{(1)}$ is the value of ρ_0/σ correct to order S^{-1}. The first equation, which says that the equality holds within lowest Born approximation, follows from Eqs. (H3a) and (C12a). To verify the second equation we use Eq. (H3) and also some extensions thereof:

$$\Psi^{(1)}_{\vec{1}\vec{2}\vec{5}\vec{6}} = \Phi^{(1)}_{\vec{5}\vec{6}\vec{1}\vec{2}} + (\hat{\omega} + \epsilon_{\vec{2}} - \epsilon_{\vec{5}} - \epsilon_{\vec{6}})(1 - x_{\vec{1}}x_{\vec{2}}x_{\vec{5}}x_{\vec{6}}) \;, \tag{I30a}$$

$$\Psi^{(7)}_{\vec{1}\vec{2}\vec{5}\vec{6}} = \Phi^{(8)}_{\vec{5}\vec{6}\vec{1}\vec{2}} + (\hat{\omega} + \epsilon_{\vec{2}} + \epsilon_{\vec{5}} + \epsilon_{\vec{6}})(x_{\vec{5}}x_{\vec{6}} - x_{\vec{1}}x_{\vec{2}}) \;, \tag{I30b}$$

$$\Psi^{(3)}_{\vec{1}\vec{6}\vec{3}\vec{5}} = \Phi^{(3)}_{\vec{3}\vec{5}\vec{1}\vec{6}} + (\epsilon_{\vec{3}} + \epsilon_{\vec{5}} + \epsilon_{\vec{6}} - \hat{\omega})(x_{\vec{6}} - x_{\vec{1}}x_{\vec{3}}x_{\vec{5}}) \;, \tag{I30c}$$

$$\Psi^{(3)}_{\vec{1}\vec{6}\vec{3}\vec{5}} = \Phi^{(3)}_{\vec{3}\vec{5}\vec{1}\vec{6}} + (\epsilon_{\vec{3}} - \hat{\omega} - \epsilon_{\vec{5}} - \epsilon_{\vec{6}})(x_{\vec{5}} - x_{\vec{1}}x_{\vec{3}}x_{\vec{6}}). \tag{I30d}$$

To derive these relations we have set $(\rho_0/\sigma) = 1$, which is correct to the order in $1/S$ to which we work.

The calculation is straightforward, but lengthy, and since it proceeds along the lines of Appendix H, it will not be given here. We quote the results on shell, i.e., when $\hat{\omega} + \epsilon_{\vec{2}} = \epsilon_{\vec{3}} + \epsilon_{\vec{4}}$:

$$(\bar{S}_{\vec{1}\vec{2}\vec{3}\vec{4}} - \Psi^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}}) - (S_{\vec{3}\vec{4}\vec{1}\vec{2}} - \Phi^{(1)}_{\vec{3}\vec{4}\vec{1}\vec{2}}) = \xi_2(\hat{\omega} - \epsilon_{\vec{1}})$$

$$\times(1 - x_{\vec{1}}x_{\vec{2}}x_{\vec{3}}x_{\vec{4}}) \;, \tag{I31a}$$

$$\bar{U}_{\vec{1}\vec{2}\vec{3}\vec{4}} - U_{\vec{3}\vec{4}\vec{1}\vec{2}} = -\xi_2(\hat{\omega} - \epsilon_{\vec{1}})(1 - x_{\vec{1}}x_{\vec{2}}x_{\vec{3}}x_{\vec{4}}) \quad , \tag{I31b}$$

where

$$\xi_2 = (NS)^{-1} \textstyle\sum_{\vec{p}} l_{\vec{p}}^2 x_{\vec{p}} \gamma_{\vec{p}} \quad , \tag{I32a}$$

$$= (2NS)^{-1} \textstyle\sum_{\vec{p}} (\epsilon_{\vec{p}}^{-1} - \epsilon_{\vec{p}}) \quad . \tag{I32b}$$

Substituting these evaluations into Eq. (I29), we see that the left-hand side of that equation is proportional to

$$[-\xi_2 + (\rho_0/\sigma)^{(1)} - 1] \equiv K_0 \quad . \tag{I33}$$

From Oguchi's work[48] we have, to first order in S^{-1}

$$\sigma = 1 - (NS)^{-1} \textstyle\sum_{\vec{p}} m_{\vec{p}}^2 \quad , \tag{I34a}$$

$$\sigma = 1 - (2NS)^{-1} \textstyle\sum_{\vec{p}} (\epsilon_{\vec{p}}^{-1} - 1) \quad , \tag{I34b}$$

and

$$\rho_0 = 1 + (2NS)^{-1} \textstyle\sum_{\vec{p}} (1 - \epsilon_{\vec{p}}) \tag{I35}$$

so that

$$(\rho_0/\sigma)^{(1)} = 1 + (2NS)^{-1} \textstyle\sum_{\vec{p}} [(1 - \epsilon_{\vec{p}}) + (\epsilon_{\vec{p}}^{-1} - 1)] \quad , \tag{I36a}$$

$$(\rho_0/\sigma)^{(1)} = 1 + (2NS)^{-1} \textstyle\sum_{\vec{p}} (\epsilon_{\vec{p}}^{-1} - \epsilon_{\vec{p}}) = 1 + \xi_2 \quad , \tag{I36b}$$

and therefore $K_0 = 0$. Thus $Q_A = Q_B$ and Eq. (I15) is verified to order S^{-2} for the processes we have considered.

In order to check Eq. (I16), we need a second-order perturbation-theory evaluation of $\Lambda'_{\alpha\alpha}$ and $\partial\Sigma'_{\alpha\alpha}/\partial\omega$. For the diagrams of Fig. 2(c), we find (omitting irrelevant constants)

$$C_6 = \Lambda'_{\alpha\alpha}(0,0) - \sigma \sim \lim_{k\to 0} \sum_{\vec{p}\vec{r}} (\epsilon_{\vec{p}}\epsilon_{\vec{r}}\epsilon_{\vec{s}}\epsilon_{\vec{k}})^{-1} \times \hat{\Phi}^{(2)}_{\vec{k}\vec{p}\vec{r}\vec{s}} \Phi^{(3)}_{\vec{r}\vec{s}\vec{k}\vec{p}} (\epsilon_{\vec{p}} + \epsilon_{\vec{r}} + \epsilon_{\vec{s}})^{-1} \quad , \tag{I37}$$

$$C_4 = \partial\Sigma'_{\alpha\alpha}(0,\omega)/\partial\omega \,|_{\omega=0} \sim \lim_{k\to 0} \sum_{\vec{p}\vec{r}} (\epsilon_{\vec{p}}\epsilon_{\vec{r}}\epsilon_{\vec{s}}\epsilon_{\vec{k}})^{-1} \times \Phi^{(2)}_{\vec{k}\vec{p}\vec{r}\vec{s}} \Phi^{(3)}_{\vec{r}\vec{s}\vec{k}\vec{p}} (\epsilon_{\vec{p}} + \epsilon_{\vec{r}} + \epsilon_{\vec{s}})^{-2} \quad . \tag{I38}$$

Since we are evaluating C_6 and C_4 to lowest order in S^{-1}, we may set $\sigma = 1$ in Eq. (I16), which then reads

$$C_6 + C_4 = 0 \quad . \tag{I39}$$

This relation will hold for the contributions of Fig. 2(c) if

$$\hat{\Phi}^{(2)}_{\vec{k}\vec{p}\vec{r}\vec{s}} (\epsilon_{\vec{p}} + \epsilon_{\vec{r}} + \epsilon_{\vec{s}}) + \Phi^{(2)}_{\vec{k}\vec{p}\vec{r}\vec{s}} = 0 \quad , \quad k \to 0 \quad . \tag{I40}$$

But according to Eq. (A9) and (C12b) this relation is valid. Accordingly, we have verified that Eq. (I16) holds when C_6 and C_4 are computed from a particular second-order diagram.

In this Appendix, we have extended the microscopic derivation of hydrodynamics at low temperatures to arbitrary order in z^{-1} and S^{-1}. The weakest point in our argument is that we have not proved the necessary conditions for hydrodynamics, Eqs. (I15) and (I16). We have made some nontrivial checks of these relations, however, and find that they are valid within low-order perturbation theory.

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*Work done at the University of Pennsylvania supported in part by the Office of Naval Research and the National Science Foundation.

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[39] Throughout this paper we are only dealing with fluctuations of the order parameter transverse to its direction of alignment, i.e., we neglect possible modes due to longitudinal fluctuations. Although such modes might exist, we believe that their effect on the transverse fluctuations (spin waves) may be neglected, in the limit of long wavelengths and low temperatures. This point is discussed in Appendix G2.

[40] This correspondence between spin waves and first sound is in fact a slight oversimplication, valid only at low temperatures, because in liquid helium there are two propagating hydrodynamic modes, first and second sound. Both of these modes involve fluctuations in the order parameter transverse to its direction of alignment. The contribution of the second-sound mode is negligible at low temperatures, but is dominant near T_c. As mentioned in Refs. 11 and 13, a system which is more closely analogous to the antiferrogmagnet is helium in fine pores. Here there is a single propagating hydrodynamic mode, fourth sound, which exhausts the order-parameter fluctuations at long wavelengths for all temperatures below T_c, and which reduces to phononlike elementary excitations at low temperatures.

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[55] The Dyson-Maleev transformation is a linear transformation which preserves the properties of addition and multiplication of operators, the commutation relations, etc. It is not a unitary transformation, however, and it therefore does not preserve the adjoint relation between operators.

[56] In writing down the Dyson-Maleev transformation for the spin operators in Eq. (2.2) we could have placed the cubic term in S_j^- rather than S_j^+. In that case, however, the Hamiltonian would contain terms of sixth order in the boson operators.

[57] The time-ordered Green's functions we introduce below are linear combinations of functions of the form of Eq. (2.4).

[58] The same type of approximation must also be made when the Holstein-Primakoff formalism is used. Moreover, in that formalism treatments of the square roots seem to involve further assumptions about the convergence of the expansion introduced.

[59] This would imply that corrections to the Fourier coefficients of the Green's functions (i.e., for discrete imaginary frequencies) are also of order $e^{-T_c/T}$. The assertion that corrections are similarly small *after* one performs the analytic continuation to the real frequency axis, involves an additional assumption. The validity of the boson formalism has been discussed by M. Wortis, Phys. Rev. 138, A1126 (1965); J. Zittartz, Z. Physik 84, 506 (1965); J. I. Davis, Ann. Phys. (N. Y.) 58, 529 (1970); J. F. Cooke and H. H. Hahn, Phys. Rev. 184, 509 (1969); Phys. Rev. B 1, 1243 (1970).

[60] The infinitesimal staggered magnetic field h necessary to align the ground state of the spin system can now be dropped, because this alignment is already implied by the use of Eq. (2.2) together with the assumption of a low density of excitations. We shall also suppress the tildes on the operators in the Dyson-Maleev representation.

[61]As we shall explain in Sec. V, the numerical coefficients appearing in our results for the damping constants are the first term in a power series in z^{-1}. Within this accuracy, it is permissible to ignore those quadratic terms arising from normally ordering the perturbative terms which are quartic in the normal-mode operators. In the context of perturbation theory, these terms contribute to $\Sigma'_{\mu\nu}(\vec{k}, \omega)$ and are included in the more general treatments in Appendixes H and I. For zero anisotropy, these terms merely scale the unperturbed quadratic spin-wave spectrum by a constant factor. For nonzero anisotropy the situation is more complicated, because these quadratic perturbations affect the transformation to normal modes. In this case, careful treatment involving renormalized potentials (see Refs. 9 and 62 and Sec. VIII) is required to produce a spectrum which does not have a spurious anisotropy gap for spin $\frac{1}{2}$. These effects are not of interest to us here because they are of higher order in z^{-1} and hence the aforementioned quadratic terms are omitted in Eq. (2.16). On the other hand, for calculations of effects of higher order in z^{-1}, as in Appendixes H and I, we take the perturbation to be the normally ordered form of V_{DM} in Eq. (2.17b), and retain the contributions of the quadratic terms to $\Sigma'_{\mu\nu}(\vec{k}, \omega)$.

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[67]By ReF and ImF we shall mean the real and imaginary parts of the function F evaluated just below the real axis as in Eq. (2.27). Where there is no risk of confusion we will use the notation F' and F'' for ReF and ImF, respectively.

[68]We adopt the convention that Greek indexes range over the values of α and β whereas Roman indexes range over the values a and b.

[69]Terms with a single vertex contribute only to $\Sigma'_{\mu\nu}(\vec{k}, E_{\vec{k}})$. The effect of these and high-order contributions to $\Sigma'_{\mu\nu}(\vec{k}, E_{\vec{k}})$ is a slight renormalization of the unperturbed energy spectrum, assuming of course, that $\Sigma'_{\mu\nu}(\vec{k}, E_{\vec{k}}) \ll E_{\vec{k}}$.

[70]For notational convenience, we shall use interchangeably the following expressions for Φ: $\Phi^{(i)}_{\vec{1}\vec{2}\vec{3}\vec{4}} \equiv \Phi^{(i)}_{\vec{1},\vec{2},\vec{3},\vec{4}} \equiv \Phi^{(i)}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)$.

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[72]By "on-resonance" we mean that the incoming magnon has $\hbar\omega = E_{\vec{k}}$. By "on-shell" we mean that intermediate magnons are undamped, i.e., are themselves on resonance. The "on-resonance" "on-shell" case $(\epsilon_{\vec{1}} + \epsilon_{\vec{2}} = \epsilon_{\vec{3}} + \epsilon_{\vec{4}})$ will often be referred to simply as "on-shell."

[73]A diagram such as in Fig. 2(c), in which an incoming magnon of energy $\hbar\omega$ breaks up into three magnons, will have negligible phase space, since the energies of all three magnons must be less than $\hbar\omega$, and their momenta are thus at most of order $\bar{\omega}$, when $\omega \to 0$.

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[75]R. J. Elliott, M. F. Thorpe, G. F. Imbusch, R. Loudon, and J. B. Parkinson, Phys. Rev. Letters 21, 147 (1968).

[76]P. A. Fleury, Phys. Rev. Letters 21, 151 (1968).

[77]Y. V. Gulayev, Zh. Eksperim. i Teor. Fiz. Pis'ma Redaktsiyu 2, 1 (1965) [Soviet Phys. JETP Letters 2, 1 (1965)].

[78]R. N. Gurzhi, Fiz. Tverd. Tela 7, 3515 (1965) [Soviet Phys. Solid State 7, 2838 (1966)].

[79]G. Reiter, Phys. Rev. Letters 20, 1170 (1968); Phys. Rev. 175, 631 (1968).

[80]Indeed, in view of the weakness of the magnon-magnon interactions (see Sec. V), we do not expect the occurrence of bound states at long wavelengths. The second magnon, on the other hand, could exist at long wavelengths as a resonance in the Green's function G_{zz}, describing longitudinal fluctuations of the order parameter (see Refs. 77–79). From phase space considerations, however, it is expected that the coupling of second magnons to the spin-wave modes at long wavelengths will be of higher order in the temperature. We shall consequently neglect the effect of second magnons on the spin-wave lifetime. The second magnon pole, of course, does not appear directly in the Green's function $G_{\alpha\alpha}$, due to the symmetry of the spin-spin interaction. This is in contrast to the case of phonons in a crystal, where the cubic anharmonic interaction couples second sound into the single-particle Green's function (see Ref. 37). We shall return briefly to the question of the coupling of spin waves to longitudinal fluctuations in Sec. V B and Appendix G 2.

[81]Because of the nonhermitian character of the Hamiltonian the quantity $\bar{\omega}_i A(\vec{k}_i, \bar{\omega}_i)$ is not everywhere non-negative. Near resonance, however, i.e., for $\bar{\omega}_i \approx \epsilon_1$, one must always have $A(\vec{k}_i, \bar{\omega}_i) > 0$. This point is discussed in Refs. 6, 7, and 82, and also in Sec. VII.

[82]R. S. Silberglitt, Ph.D. thesis, University of Pennsylvania, 1968 (unpublished).

[83]In fact, for the derivation of Appendix D, it is only necessary that the first and third moments of $\varphi(x)$ vanish.

[84]In Appendix G 2, we examine a typical ladder. A physically meaningful resummation would have to include α and β magnons on an equal footing in order to describe properly the longitudinal *spin* fluctuations. At the very least, this would require resummation of general (i.e., involving α and β magnons in all intermediate states) particle-hole ladders, rather than just the α-particle α-hole ladders discussed in Appendix G 2.

[85]There are slight modifications which must be made to enumerate diagrams properly. This point is discussed in Appendix G 3.

[86]This can be seen from the equality of $\psi^{+(i)}_{\vec{1}\vec{2}\vec{3}}$ and $\psi^{-(i)}_{\vec{1}\vec{2}\vec{3}}$ in Eq. (19) of Ref. 62. For the anisotropic case, in order to obtain this result one must also renormalize the terms proportional to ϵ_4^2 in Eq. (17) of Ref. 62.

[87]F. Schwabl and K. H. Michel, Solid State Commun. 7, 1781 (1969); Z. Physik 238, 264 (1970).

[88]These series consist of terms of the type $s^{-n}z^{-m}$ with $m \geq n$. Sometimes it is convenient to regard them as series in S^{-1}, as in Ref. 48. In this case the coefficient of S^{-n} is an infinite series in z^{-1}. To reproduce kinematic properties which are connected with the finiteness of S, or for the anisotropic system, it is essential to consider

z^{-1} to be the expansion parameter, as is done in Refs. 9 and 62. In this case, the coefficient of z^{-m} is an mth-order polynomial in S^{-1}. The z^{-1} expansion has been discussed from a more general point of view by the authors cited in Refs. 28 and 29.

[89]Since Eqs. (G34) are not positive definite forms in the matrix elements, it is conceivable that corrections in the vertex functions might cancel when the decay rate is calculated.

[90]Numerical factors and occupation numbers are omitted here for simplicity, as they are not essential for the present discussion.

[91]V. N. Kashcheev and V. N. Krivoglaz, Fiz. Tverd. Tela 3, 1541 (1961) [Soviet Phys. Solid State 3, 1117 (1961)].

[92]P. D. Loly, Ann. Phys. (N. Y.) 56, 40 (1970). See also P. D. Loly and P. Mikušik, Phys. Rev. B 1, 3204 (1970).

[93]The results for the spin spectral-weight function should be independent of the boson representation employed, even though this property is not shared by the boson Green's functions. In Appendix F, we substantiate this statement by using the Holstein-Primakoff formalism in the regime $\epsilon_{\vec{k}} \gg \gamma$ to reproduce the result for Im $\mathcal{G}$ obtained here. For $\epsilon_{\vec{k}} \ll \gamma$ we do not have self-consistent expressions for the Green's functions in the Holstein-Primakoff formalism.

[94]This result is only valid to lowest order in $\epsilon_{\vec{k}}$, τ, and $(zS)^{-1}$ (see Appendix I). The same result holds for the ferromagnet because the spin Green's function of Refs. 6 and 7, $\mathcal{G} \approx [1+(2S)^{-1}\Lambda]\,[\hbar\omega - E_{\vec{k}} - \Sigma_{\vec{k}}(\omega)]^{-1}$, differs from $[\hbar\omega - E_{\vec{k}} - \Sigma^{s}_{\vec{k}}(\omega)]^{-1}$ by terms of second order in the density of spin deviations [here $\Sigma^{s}_{\vec{k}}(\omega) = \Sigma_{\vec{k}}(\omega) + (2S)^{-1} \times (\hbar\omega - E_{\vec{k}})\, A_{\vec{k}}(\omega)$]. Such differences are beyond the accuracy of the treatments used in Refs. 6 and 7.

[95]The appearance of terms involving the discontinous factor sgn $(\rho^2 - 1)$ in Eq. (7.19) may seem rather anomalous. Note that on resonance, when $\rho^2 \approx 1$, such terms are smaller than the resonant terms by a factor $(C_1^0 + C_2^0)\,\epsilon_{\vec{k}} \ll 1$, and hence do not appear in Eq. (7.20).

[96]L. D. Landau, Zh. Eksperim. i Teor. Fiz. 30, 1058 (1956) [Soviet Phys. JETP 3, 920 (1957)].

[97]R. D. Lowde, J. Appl. Phys. 36, 884 (1965).

[98]See, for instance, T. Riste, J. Phys. Soc. Japan Suppl. 17, 60 (1962); J. L. Beeby and J. Hubbard, Phys. Letters 26A, 376 (1968).

[99]The same symmetry argument shows that there is no second-magnon pole in the transverse spin-correlation function, even though such a pole might be present in other correlation functions.

[100]M. G. Cottam and R. B. Stinchcombe, J. Phys. C 3, 2326 (1970).

[101]R. B. Woolsey and R. M. White, Phys. Rev. 188, 813 (1969).

[102]F. M. Johnson and A. H. Nethercot, Jr., Phys. Rev. 114, 705 (1959).

[103]F. Keffer, Phys. Rev. 87, 608 (1952).

[104]In addition to the uncertainties due to our treatment of the dipolar interactions using a single-ion anisotropy term, our model for MnF_2 is also rather crude in that we have described the exchange interactions by means of a single exchange integral, as if the crystal structure were cubic rather than tetragonal, as is actually the case.

[105]M. S. Seehra and T. G. Castner, Jr., Phys. Rev. B 1, 2289 (1970).

[106]In fact the $\Phi^{(i)}$ should be replaced by $H_E^{-1}\,\hat{\Phi}^{(i)}$. In order to keep the matrix elements dimensionless, we shall replace $\phi^{(i)}$ by $\hat{\phi}^{(i)}$ in the matrix elements and keep track of the factor H_E^{-1} separately. Also we should point out that in computing these matrix elements, there occurs one less factor of 2 than in Eq. (2.39), because contractions are permitted only with $\alpha_{\vec{k}}$ or $\beta^{\dagger}_{-\vec{k}}$ in Eq. (C11). Then comparison of Eqs. (C11) and (2.17b) shows that $\hat{\phi}^{(i)}$ always appears with the same factors as $\pm H_E\,\phi^{(i)}$.

[107]The correct result may be obtained quite simply by averaging the factors outside the δ function over μ. This procedure is justified by the detailed calculations in Appendix D.

[108]If we regard $\alpha_{\vec{k}}$ and $\beta_{\vec{k}}$ in Eq. (C11) as parameters, then Eq. (C20) means that $\partial V_{\mathrm{eff}}/\partial\alpha^{\dagger}_{\vec{k}} = \partial V_{\mathrm{eff}}/\partial\beta_{-\vec{k}}$. This way of writing Eq. (C20) takes proper account of the minus signs appearing in Eq. (C11).

[109]Throughout this Appendix we shall use the same notation for the various quantities in the Holstein-Primakoff formalism as we did in the Dyson-Maleev formalism, except for the potential V_{HP} in Eq. (F3) below. Whenever a confusion is considered possible, we shall explicitly insert the subscripts HP or DM to distinguish the two formalisms.

[110]The method to determine the sign of the infinitesimal imaginary part when there is no complex external energy is discussed in Ref. 54.

[111]For a discussion of the symmetry number, see T. Schultz, *Quantum Field Theory and the Many Body Problem* (Gordon and Breach, New York, 1964), p. 46.

[112]Note that we are concerned with the symmetry of the *diagrams* not of the *functions*. For example, the function $U^{(1)}_{\vec{3}\vec{4}\vec{1}\vec{2}}$ is symmetric in its first two arguments, as can be seen from Eq. (H2a). Since the scattering process described by $R^{(4)}$, for example, does not involve two equivalent particles, it is not decomposed into symmetric and unsymmetric components.

[113]This property holds for finite ω, since it depends only on the form of the external matrix elements in the limit $k \rightarrow 0$.

[114]Note that umklapp processes, i.e., processes with nonzero values of $\vec{k}_5 + \vec{k}_6 - \vec{k}_1 - \vec{k}_2$, give contributions which are of higher order in $k_B T/JS$. Also in Appendixes H and I for notational convenience we shall drop the superscript (1) on the quantities $S^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}}$ and $U^{(1)}_{\vec{1}\vec{2}\vec{3}\vec{4}}$. Our discussion of these quantities is correct only to first order in $1/S$.